if $\underline{a} \in V$ then $\gamma\underline{a} = \gamma(\alpha_1\underline{v}_1 + \cdots + \alpha_n\underline{v}_n) = (\gamma\alpha_1)\underline{v}_1 + \cdots + (\gamma\alpha_n)\underline{v}_n$ $\square$

<u>Defn</u> A set of vectors $\underline{v}_1, \ldots, \underline{v}_n$ is a <u>spanning set</u> or spans a vector space V if every vector $\underline{v} \in V$ is a linear combination of $\underline{v}_1, \ldots, \underline{v}_n$.

<u>Example</u> $\underline{i}, \underline{j}$ span $\mathbb{R}^2$ $\qquad$ $\underline{i}, \underline{j}, \underline{k}$ span $\mathbb{R}^3$

$\underline{i}, \underline{j}, \underline{i}+\underline{j}$ span $\mathbb{R}^2$

$\underline{i}, \underline{j}, \underline{i}+\underline{j}$ do not span $\mathbb{R}^3$.

<u>Defn</u> vectors $\underline{v}_1, \ldots, \underline{v}_n$ are <u>linearly dependent</u> if one vector is a linear combination of the others. They are <u>linearly independent</u> if they are not linearly dependent. [Lin. dep. equivalent: non-trivial linear combination is zero].

<u>Example</u> $(1,0,1)$ and $(2,-1,3)$ are linearly independent in $\mathbb{R}^3$.

Suppose not: then $\alpha(1,0,1) + \beta(2,-1,3) = \underline{0}$.

$$(\alpha+2\beta, -\beta, \alpha+3\beta) = 0 \quad \Rightarrow \quad \beta = 0 \Rightarrow \alpha = 0.$$

<u>Thm</u> 1) $\underline{v}_1, \ldots, \underline{v}_n$ are linearly dependent iff $\alpha_1\underline{v}_1 + \cdots + \alpha_n\underline{v}_n = \underline{0}$, not all $\alpha_i = 0$.

2) $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent iff $\alpha_1\underline{v}_1 + \cdots + \alpha_n\underline{v}_n = \underline{0} \Rightarrow$ all $\alpha_i = 0$.

<u>Proof</u> 1) $\Rightarrow$ suppose $\underline{v}_1, \ldots, \underline{v}_n$ dependent $\Rightarrow$ e.g. $\underline{v}_1 = \alpha_2\underline{v}_2 + \cdots + \alpha_n\underline{v}_n$

$\Rightarrow 1\underline{v}_1 - \alpha_2\underline{v}_2 - \alpha_3\underline{v}_3 - \cdots - \alpha_n\underline{v}_n = 0$ (not all $\alpha_i = 0$).

$\Leftarrow$ suppose $\alpha_1\underline{v}_1 + \cdots + \alpha_n\underline{v}_n = 0$ at least one $\alpha_i \neq 0$ relabel so $\alpha_1 \neq 0$.

$$\underline{v}_1 = -\frac{\alpha_2}{\alpha_1}\underline{v}_2 - \frac{\alpha_3}{\alpha_1}\underline{v}_3 - \cdots - \frac{\alpha_n}{\alpha_1}\underline{v}_n. \quad \square$$ (2 is contrapositive of 1).

<u>Example</u> Show $(1,0,3,1), (0,1,-6,-1), (0,2,1,0)$ are linearly independent.

$$c_1(1,0,3,1) + c_2(0,1,-6,-1) + c_3(0,2,1,0) = (0,0,0,0)$$

$c_1 = 0$, $\quad c_2 + 2c_3 = 0$, $\quad 3c_1 - 6c_2 + c_3 = 0$, $\quad c_1 - c_2 = 0$

$\Rightarrow c_1 = 0$ $\quad \Rightarrow c_3 = 0$ $\qquad\qquad c_1 = 0 \Rightarrow c_2 = 0$.

Def^n A **basis** for a vector space is a set of vectors which **span** and are **linearly independent**.

Examples. $\mathbb{R}^2$ has basis $\underline{i}, \underline{j}$, also $2\underline{i}$, $\underline{i}+\underline{j}$

$\mathbb{R}^3$ has basis $\underline{i}, \underline{j}, \underline{k}$. also $\underline{i}+\underline{j}$, $\underline{i}-\underline{j}$ $\underline{i}+\underline{j}+\underline{k}$.

$\mathbb{R}^n$ has basis $\underline{e}_1, \dots, \underline{e}_n$ where $\underline{e}_1 = \langle 1, 0, \dots, 0 \rangle$

$\underline{e}_i = \langle 0, \dots 1, \dots 0 \rangle$ ← ith place.

Note a set of vectors containing $\underline{0}$ is not L.I!

Non-examples $\underline{i}$ is not a basis for $\mathbb{R}^2$ (doesn't span)

$\underline{i}, \underline{j}, \underline{i}+\underline{j}$ not a basis for $\mathbb{R}^2$. (not linearly independent)

Thm let $V \subset \mathbb{R}^n$ be a vector subspace. If V is spanned by $\underline{v}_1, \dots, \underline{v}_n$ then there is a subset $\underline{v}_{i_1} \dots \underline{v}_{i_k}$ which is a basis.

Proof (sketch) if $\underline{v}_1, \dots, \underline{v}_n$ linearly independent, then done.

if not, some $\underline{v}_i = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$ not all $\alpha_i = 0$.

reorder $\underline{v}_n = \alpha_1 \underline{v}_1 + \dots + \alpha_{n-1} \underline{v}_{n-1}$, then $\underline{v}_1, \dots \underline{v}_{n-1}$ span. V

continue by induction. □

Thm Let $V \subset \mathbb{R}^n$ be a vector subspace. Let $\underline{s}_1, \dots \underline{s}_m$ span V
$\underline{b}_1, \dots \underline{b}_n$ be a basis for V

then $n \le m$.

Proof $\underline{s}_i$ span, so $\underline{b}_1 = \alpha_1 \underline{s}_1 + \dots + \alpha_n \underline{s}_n$ for some α_i not all zero.

reorder may assume $\alpha_1 \ne 0$.

consider $\underline{b}_1, \underline{s}_2, \dots \underline{s}_m$. **claim** this is still a spanning set.

$\underline{b}_1 = \alpha_1 \underline{s}_1 + \dots + \alpha_m \underline{s}_m$ $\alpha_1 \ne 0 \Rightarrow$ $\underline{s}_1 = \frac{1}{\alpha_1} \underline{b}_1 - \frac{\alpha_2}{\alpha_1} \underline{s}_2 - \dots - \frac{\alpha_m}{\alpha_1} \underline{s}_m$

so if $\underline{v} = \beta_1 \underline{s}_1 + \beta_2 \underline{s}_2 + \dots + \beta_m \underline{s}_m$ then $\underline{v} = \beta_1' \underline{b}_1 + \beta_2' \underline{b}_2 + \dots + \beta_m' \underline{b}_m$

continue — can put all $\underline{b}_i$'s into spanning set $\underline{b}_1, \dots \underline{b}_n, \underline{s}_{n+1} \cdots \underline{s}_m$
$\Rightarrow m \geq n$.

— stop with spanning set $\underline{b}_1, \dots, \underline{b}_r$ $\quad r < n$
but then $\underline{b}_{r+1} = \alpha_1 \underline{b}_1 + \dots + \alpha_r \underline{b}_r$ # linear independence
$\square$.

Corollary Let $V \subset \mathbb{R}^n$ be a vector space with bases $\underline{a}_1, \dots \underline{a}_r$, and $\underline{b}_1, \dots \underline{b}_s$, then $\underline{r = s}$.

Proof $r \leq s$ and $s \leq r$! $\square$.

Def$^{\underline{n}}$ The number of vectors in a basis is the dimension of the vector space.

Example $\mathbb{R}^2$ has bases $\{\underline{i}, \underline{j}\}$ and $\{\underline{i}, \underline{i}+\underline{j}\}$ etc.

coordinates let $\underline{v} \in V$, with basis $\underline{b}_1, \dots \underline{b}_n$

then $\underline{v} = c_1 \underline{b}_1 + c_2 \underline{b}_2 + \dots + c_n \underline{b}_n$

$(c_1, \dots, c_n)$ are the coordinates of $\underline{v}$ wrt the basis $\underline{b}_1, \dots, \underline{b}_n$.

claim $(c_1, \dots, c_n)$ are unique, given $\underline{v}$ and $\underline{b}_1, \dots, \underline{b}_n$

Proof suppose $c_1 \underline{b}_1 + \dots + c_n \underline{b}_n = d_1 \underline{b}_1 + \dots + d_n \underline{b}_n$
then $(c_1 - d_1) \underline{b}_1 + \dots + (c_n - d_n) \underline{b}_n = \underline{0} \Rightarrow c_i - d_i = 0$ for all i
$\Rightarrow c_i = d_i$ $\square$.

Example $\mathbb{R}^2$ has basis $\{\underline{i}, \underline{j}\}$ ① $\{\underline{i}+\underline{j}, \underline{i}-\underline{j}\}$ ②

$\underline{v} = \langle 1,2 \rangle = \underline{i} + 2\underline{j}$ in ①

② ? $\langle 1,2 \rangle = a \langle 1,1 \rangle + b \langle 1,-1 \rangle$ ① $a + b = 1$ ② $a - b = 2$

$b = -\frac{1}{2}$

$a = \frac{1}{2}$

check!

$\frac{3}{2} \langle 1,1 \rangle - \frac{1}{2} \langle 1,-1 \rangle = \langle 1,2 \rangle$.

special bases : orthogonal bases : $\underline{b}_1, \dots \underline{b}_n$ with $\underline{b}_i \cdot \underline{b}_j = 0$ if $i \neq j$

note $\underline{i}, \underline{j}$ is orthogonal. $\underline{i} \cdot \underline{j} = 0$

$\underline{i}, \underline{i}+\underline{j}$ not orthogonal $\underline{i} \cdot (\underline{i}+\underline{j}) = 1$.

$\underline{v} \in V$ $\underline{b}_1, \dots, \underline{b}_n$ orthogonal basis.

then $\underline{v} = \alpha_1 \underline{b}_1 + \dots + \alpha_n \underline{b}_n$

$$\underline{v} \cdot \underline{b}_i = \alpha_1 \underline{b}_1 \cdot \underline{b}_i + \dots + \alpha_i \underline{b}_i \cdot \underline{b}_i + \dots + \alpha_n \underline{b}_n \cdot \underline{b}_i$$

$$= \alpha_i \|\underline{b}_i\|^2$$

$$\alpha_i = \frac{\underline{v} \cdot \underline{b}_i}{\|\underline{b}_i\|^2}.$$

orthonormal basis $\underline{b}_1, \dots \underline{b}_n$ $\underline{b}_i \cdot \underline{b}_j = 0$ $i \neq j$ $\|\underline{b}_i\|^2 = 1$ for all i.

Gram – Schmidt

Thm Given a basis $\underline{a}_1, \dots \underline{a}_n$ we can make an orthogonal basis $\underline{b}_1, \dots, \underline{b}_n$

Proof set $\underline{b}_1 = \underline{a}_1$

set $\underline{b}_2 = \underline{a}_2 + c\underline{a}_1$: choose c s.t. $\underline{b}_1, \underline{b}_2$ orthogonal.

i.e. want $\underline{b}_1 \cdot \underline{b}_2 = 0 = \underline{a}_1 \cdot (\underline{a}_2 + c\underline{a}_1)$

$= \underline{a}_1 \underline{a}_2 + c\, \underline{a}_1 \cdot \underline{a}_1 \Rightarrow c = -\dfrac{\underline{a}_1 \cdot \underline{a}_2}{\underline{a}_1 \cdot \underline{a}_1}$

so $\underline{b}_2 = \underline{a}_2 - \dfrac{\underline{a}_1 \cdot \underline{a}_2}{\|\underline{a}_1\|^2} \underline{a}_1$

now set $\underline{b}_3 = \underline{a}_3 + c_1 \underline{b}_1 + c_2 \underline{b}_2$

want $\underline{b}_1 \cdot \underline{b}_3 = 0 = \underline{b}_1 \cdot \underline{a}_3 + c_1 \underline{b}_1 \cdot \underline{b}_1 + c_2 \cancel{\underline{b}_1 \cdot \underline{b}_2}^{0}$ $\quad c_1 = \dfrac{\underline{a}_3 \cdot \underline{b}_1}{\underline{b}_1 \cdot \underline{b}_1}$

$\underline{b}_2 \cdot \underline{b}_3 = 0 = \underline{b}_2 \cdot \underline{a}_3 + c_1 \cancel{\underline{b}_2 \cdot \underline{b}_1}^{0} + c_2 \underline{b}_2 \cdot \underline{b}_2$ $\quad c_2 = \dfrac{\underline{a}_3 \cdot \underline{b}_2}{\underline{b}_2 \cdot \underline{b}_2}$ etc...

§7.1 Matrices

Defn A matrix is a rectangular array of numbers.

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $[a_{ij}]$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$. (2×3) matrix

$(n\times m)$ matrix rows x columns

two matrices are the same if they have equal entries ($\Rightarrow$ and same dimension)

Addition $\underset{(n\times m)}{A} + \underset{(n\times m)}{B} = C$ $[c_{ij}] = [a_{ij} + b_{ij}]$.

Scalar multiplication $\lambda A = [\lambda a_{ij}]$

Note: the set of all $(n\times m)$ matrices is a vector space.

Q: what is its dimension?

Matrix multiplication $\underset{(r\times s)}{A}\ \underset{(s\times t)}{B} = \underset{(r\times t)}{C}$ $[c_{ij}] =$ "$\text{row}_i \cdot \text{column}_j$"

Example $\underset{(2\times 3)}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} \underset{(3\times 2)}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{(2\times 2)}{\begin{bmatrix} 0 & 5 \\ 4 & 3 \end{bmatrix}}$

$A = [a_{ij}]$ $B = [b_{ij}]$

$AB = C = [c_{ij}]$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{is}b_{sj}$

$= \sum_{k=1}^{s} a_{ik}b_{kj}$

note $\underset{(2\times 3)}{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$ doesn't work! $3 \neq 2$.

so $AB \neq BA$ in general!

useful properties

1. $A+B = B+A$
2. $A(B+C) = AB+AC$ (order matters!)
3. $(A+B)C = AC+BC$
4. $(AB)C = A(BC)$ (associative)
5. $(cA)B = c(AB) = A(cB)$