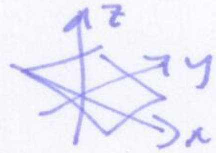
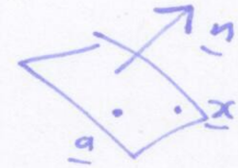
Equations of lines: 

$L(t) = L: \underline{r}(t) = \underline{a} + t\underline{v}$

warning parameterizations are not unique $L: \underline{r}(t) = (\underline{a} - \underline{v}) + t(-\tfrac{1}{2}\underline{v})$

Equations of planes: $ax + by + cz = d.$





$(\underline{x} - \underline{a}) . \underline{n} = 0$

$\underline{x} = \langle x, y, z \rangle$
$\underline{n} = \langle a, b, c \rangle$
$\underline{a} = \langle r, s, t \rangle.$

$\langle x - r, y - s, z - t \rangle . \langle a, b, c \rangle = \quad ax + by + cz = \underbrace{ar + bs + ct}_{d}.$

Example find where $x + 2y - z = 3$ intersects $\langle 1,2,3 \rangle + t\langle 1,-1,-2 \rangle$.

Q: do the two lines $\langle 1,0,-1 \rangle + t\langle -1,2,1 \rangle$
$\langle 2,1,2 \rangle + t\langle 1,-1,-1 \rangle$ intersect? solve: $1 - t = 2 - s$
$+2t = 1 + s$
$-1 + t = 2 - s.$

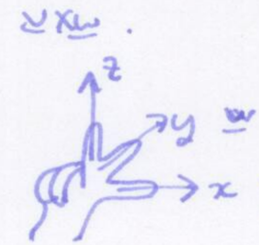
§6.3 Cross products

$\underline{v}, \underline{w}$ vectors $\underline{v} \times \underline{w}$ is another vector.

Def^{n} (geometric) $\|\underline{v} \times \underline{w}\| = \|\underline{v}\| \|\underline{w}\| |\sin\theta|$

$\underline{v} \times \underline{w}$ is perpendicular to both $\underline{v}, \underline{w}$.

$\underline{v}, \underline{w}, \underline{v} \times \underline{w}$ form a right handed triple



recall $\det(A)$ A 3x3 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Def^{n} (components) $\underline{v} = \langle v_1, v_2, v_3 \rangle$
$\underline{w} = \langle w_1, w_2, w_3 \rangle$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

useful properties

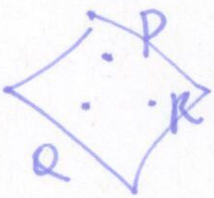
1. $\underline{v}\times\underline{w} = -\underline{w}\times\underline{v}$ (antisymmetric / anti-commutative)

2. $\underline{u}\times(\underline{v}+\underline{w}) = \underline{u}\times\underline{v} + \underline{u}\times\underline{w}$
   $(\underline{u}+\underline{v})\times\underline{w} = \underline{u}\times\underline{w} + \underline{v}\times\underline{w}$ } distribution, order matters!

3. $c(\underline{u}\times\underline{v}) = (c\underline{u})\times\underline{v} = \underline{u}\times(c\underline{v})$.

Remark $\underline{u}\cdot(\underline{u}\times\underline{v}) = 0$. (check!)

Example find plane through $(1,1,2)$, $(1,3,4)$ $(-1,-2,1)$
P Q R



$\overrightarrow{PQ} = \langle 0,2,2\rangle$
$\overrightarrow{PR} = \langle -2,-3,-1\rangle$ $\qquad \underline{n} = \overrightarrow{PQ}\times\overrightarrow{PR}$

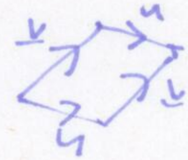
$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & 2 \\ -2 & -3 & -1 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 2 \\ -3 & -1\end{vmatrix} - \underline{j}\begin{vmatrix} 0 & 2 \\ -2 & -1\end{vmatrix} + \underline{k}\begin{vmatrix} 0 & 2 \\ -2 & -3\end{vmatrix}.$$

$$= \langle 4, -4, 4\rangle$$

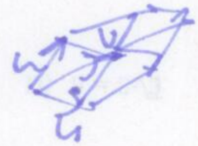
$(\underline{a}-\underline{x})\cdot\underline{n} = 0$ $\qquad (\langle 1,1,2\rangle - \langle x,y,z\rangle)\cdot(1,-1,1) = 0$.

useful property

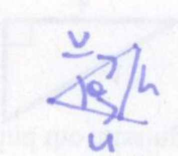
4. $\|\underline{u}\times\underline{v}\| =$ area of parallelogram



5. $|\underline{u}\cdot(\underline{v}\times\underline{w})| =$ volume of parallelepiped

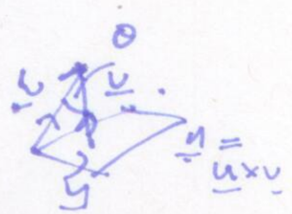


Proof (4)



area of parallelogram = base × height
$\|\underline{u}\|\,\|\underline{v}\|\sin\theta$

(5) volume of parallelogram = $\overset{\text{area of}}{\text{base}}$ × height
$\|\underline{u}\times\underline{v}\|\cos\theta\,\|\underline{w}\|$.



§6.4 The vector space $\mathbb{R}^n$

Def$^{\underline{n}}$ $\mathbb{R}^n$: set of all n-vectors: n-tuples $(x_1, x_2, \dots x_n)$

addition: $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1+y_1, x_2+y_2 + \dots + x_n+y_n)$

scalar multiplication: $c(x_1, \dots, x_n) = (cx_1, \dots, cx_n)$

special vectors: zero vector $\underline{0} = (0, 0, \dots, 0)$

$-\underline{x} = (-x_1, -x_2, \dots, -x_n)$.

Useful properties

1. $\underline{a} + \underline{b} = \underline{b} + \underline{a}$
2. $\underline{a} + (\underline{b} + \underline{c}) = (\underline{a} + \underline{b}) + \underline{c}$
3. $\underline{a} + \underline{0} = \underline{a}$
4. $(\alpha + \beta)\underline{a} = \alpha \underline{a} + \beta \underline{b}$
5. $(\alpha\beta)\underline{a} = \alpha(\beta \underline{a})$
6. $\alpha(\underline{a} + \underline{b}) = \alpha \underline{a} + \beta \underline{b}$
7. $\alpha \underline{0} = \underline{0}$

Def$^{\underline{n}}$ length / norm $\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

dot product $\underline{x} \cdot \underline{y} = x_1y_1 + x_2y_2 + \dots + x_ny_n$ so $\underline{x} \cdot \underline{x} = \|\underline{x}\|^2$.

useful properties

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$
2. $(\underline{a} + \underline{b}) \cdot \underline{c} = \underline{a} \cdot \underline{c} + \underline{b} \cdot \underline{c}$
3. $\alpha(\underline{a} \cdot \underline{b}) = (\alpha \underline{a}) \cdot \underline{b} = \underline{a} \cdot (\alpha \underline{b})$
4. $\underline{a} \cdot \underline{a} = 0 \Rightarrow \underline{a} = \underline{0}$
5. $\|\alpha \underline{a} + \beta \underline{b}\|^2 = \alpha^2 \|\underline{a}\|^2 + 2\alpha\beta\, \underline{a} \cdot \underline{b} + \beta^2 \|\underline{b}\|^2$
6. (Cauchy Schwartz) $|\underline{a} \cdot \underline{b}| \leq \|\underline{a}\| \|\underline{b}\|$.
8. $\underline{a} \cdot \underline{b} = \|\underline{a}\| \|\underline{b}\| \cos\theta$.

<u>Def^n</u> A subset $S \subset \mathbb{R}^n$ is a <u>subspace</u> / <u>subvector space</u> / <u>vector space</u> if

- $\underline{0} \in S$
- sum of any two vectors in S is a vector in S (additive closure)
- any scalar multiple of a vector in S is in S (closed under scalar multiplication).

<u>equivalently</u> for all $s, t \in S$, $\alpha s + \beta t \in S$.

<u>Examples</u> $\underline{0} \in \mathbb{R}^n$ solution to $x+y+z=0$ in $\mathbb{R}^3$.

$\{(x, y, 0) \in \mathbb{R}^n \mid x \in \mathbb{R}, y \in \mathbb{R}\}$. all multiples of $(1,2,3,4)$ in $\mathbb{R}^4$.

<u>Non-examples</u> unit vectors. solution to $x+y+z=1$ in $\mathbb{R}^3$.
~~pp~~ vectors with positive entries

<u>Subspaces of $\mathbb{R}^2$</u>

- $\mathbb{R}^2$
- $\underline{0}$
- lines through the origin. (scalar multiples of a specific vector).

<u>Subspaces of $\mathbb{R}^3$</u>

- $\mathbb{R}^3$, $\cdot\ \underline{0}$ $\cdot$ lines thru origin, planes through origin.

<u>§6.5 Linear independence, spanning sets, dimension</u>

<u>Def^n</u> a linear combination of vectors $\underline{v}_1, \dots, \underline{v}_n$ is a sum

$$\alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \dots + \alpha_n \underline{v}_n, \text{ each } \alpha_i \in \mathbb{R}.$$

<u>Example</u>. $2(1,-1,0,2) - 3(2,3,1,1)$

<u>Thm</u> The set of all linear combinations of a collection of vectors is a vector space.

<u>Proof</u> Let $V =$ all linear combinations of $\underline{v}_1, \dots, \underline{v}_n$

$\underline{0} \in V$. clear $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.

If $\underline{a}, \underline{b} \in V$ then $\underline{a} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$ $\quad \underline{a}+\underline{b} = (\alpha_1+\beta_1)\underline{v}_1 + \dots + (\alpha_n+\beta_n)\underline{v}_n$

$\underline{b} = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$