Examples

overdamped $\qquad y''+6y'+5y = 6\sqrt{5}\cos(\sqrt{5}\,t) \qquad y(0)=y'(0)=0$

has solution $\qquad y(t) = \frac{\sqrt{5}}{4}\left(-e^{-t}+e^{-5t}\right) + \sin(\sqrt{5}\,t)$

critically damped $\qquad y''+2y'+y = 2\cos(t) \qquad y(0)=y'(0)=0$

$$y(t) = -te^{-t} + \sin(t)$$

underdamped $\qquad y''+2y'+y = 2\sqrt{2}\cos(\sqrt{2}t) \qquad y(0)=y'(0)=0$

$$y(t) = -\sqrt{2}\,e^{-t}\sin(t) + \sin(\sqrt{2}t)$$

Resonance (suppose $c=0$) $\qquad y'' + \frac{k}{m}y = \frac{A}{m}\cos(\omega t)$

($\omega_0 \neq \omega$)
general solution : $\qquad y(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{A}{m(\omega_0^2-\omega^2)}\cos(\omega t)$ (*)

$\omega_0 = \sqrt{\frac{k}{m}} \leftarrow$ called natural frequency.

$\omega \leftarrow$ called input frequency.

suppose $\omega_0=\omega$ (resonant case). (*) not a solution. to $\quad y'' + \frac{k}{m}y = \frac{A}{m}\cos(\omega_0 t)$

$$y_h(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

try : $\quad y_p(t) = at\cos(\omega_0 t) + bt\sin(\omega_0 t)$

$$y_p'(t) = a\cos(\omega_0 t) - at\sin(\omega_0 t)\omega_0 + b\sin(\omega_0 t) + bt\cos(\omega_0 t)\omega_0$$

$$y_p''(t) = -a\omega_0\sin(\omega_0 t) - a\omega_0\sin(\omega_0 t) - a\omega_0^2 t\cos(\omega_0 t)$$
$$+ b\cos(\omega_0 t)\omega_0 + b\cos(\omega_0 t)\omega_0 - b\omega_0^2 t\sin(\omega_0 t).$$

plug in to (*) : $\qquad = -2a\omega_0\sin(\omega_0 t) - a\omega_0^2 t\cos(\omega_0 t)$
$$+ 2b\omega_0\cos(\omega t) - b\omega_0^2 t\sin(\omega_0 t)$$

$y'' + \frac{k}{m} y = \frac{A}{m} \cos(\omega_0 t)$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

$$\cos(\omega_0 t)\left[\cancel{-a\omega_0^2 t} + 2b\omega_0 + \cancel{\frac{k}{m} a t}\right] + \sin(\omega_0 t)\left[-2a\omega_0 \cancel{- b\omega_0^2 t + \frac{k}{m} bt}\right] = \frac{A}{m}\cos(\omega_0 t)$$

$$2b\omega_0 \cos(\omega_0 t) - 2a\omega_0 \sin(\omega_0 t) = \frac{A}{m}\cos(\omega_0 t)$$

for all t: $\quad a = 0, \quad 2b\omega_0 = \frac{A}{m}, \quad b = \frac{A}{2m\omega_0}$

so $\quad y_p(t) = \dfrac{A}{2m\omega_0}\, t \sin(\omega_0 t)$

general solution: $\quad y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega t) + \dfrac{A}{2m\omega_0}\, t \sin(\omega_0 t)$

sol^{n} $\to \infty$ as $t \to \infty$!

Beats $\quad$ ($c = 0$, no damping) $\quad y'' + \omega_0^2 y = \frac{A}{m}\cos(\omega t)$, $y(0) = 0$ $y'(0) = 0$

$\omega \neq \omega_0$

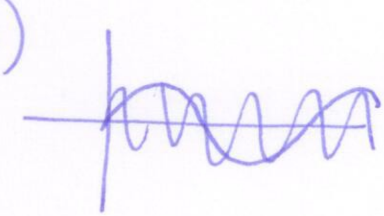
use: addition formulae:

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$.

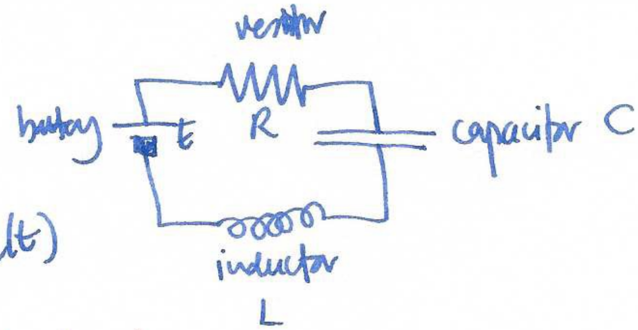
$$y(t) = \frac{A}{m(\omega_0^2 - \omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

$$= \frac{2A}{m(\omega_0^2 - \omega^2)} \sin\left(\tfrac{1}{2}(\omega_0 + \omega)t\right) \sin\left(\tfrac{1}{2}(\omega_0 - \omega)t\right)$$



get:

Electrical circuits



voltage

$$E(t) = Li'(t) + Ri(t) + \frac{1}{C}q(t)$$

$$i(t) = q'(t) \rightsquigarrow \quad q'' + \frac{R}{L}q' + \frac{1}{LC}q = \frac{1}{L}E \quad \leftarrow \text{exactly same as spring equation!}$$

analogy:

displacement function	$y(t)$	$\leftrightarrow$	charge $q(t)$	
velocity	$y'(t)$	$\leftrightarrow$	$i(t)$	current
driving force	$f(t)$	$\leftrightarrow$	$E(t)$	electromotive force
mass	m	$\leftrightarrow$	L	inductance
damping constant	c	$\leftrightarrow$	R	resistance
spring modulus	k	$\leftrightarrow$	$\frac{1}{C}$	1/capacitance

Higher order differential equations

Reduction to first order: $\quad y'' + 2y' + y = 0$ ⊛ try: $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, $y'' = \lambda^2 e^{\lambda x}$

$$e^{\lambda x}(\lambda^2 + 2\lambda + 1) = 0 \quad \text{etc...}$$

alternatively: set

$$\left.\begin{aligned} z_1 &= y \\ z_2 &= y' = z_1' \\ y'' &= z_2'. \end{aligned}\right\} \text{⊛ becomes } z_2' + 2z_2 + z_1 = 0.$$

⊛ equivalent to

$$\left.\begin{aligned} z_1' &= z_2 \\ z_2' &= -z_1 - 2z_2 \end{aligned}\right\} \text{2d first order d.e.!}$$

set $\underline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $\quad \underline{z}' = A\underline{z}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

look for solution: $\underline{z} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda x}$ $\quad \underline{z}' = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \lambda e^{\lambda x}$

$$\underline{z}' = A\underline{z} : \quad \lambda \underline{c} e^{\lambda x} = A\underline{c}e^{\lambda x} \quad \Rightarrow \quad A\underline{c} = \lambda \underline{c}$$

$\underline{c}$ eigenvector, λ eigenvalue of A.

Example $y''' + 2y'' - y' - 2y = 0$ ⊛

$z_1 = y$ $\qquad z_1' = z_2$

$z_2 = y' = z_1'$ $\qquad z_2' = z_3$ $\qquad \underline{z}' = A\underline{z}$.

$z_3 = z_2'$ $\qquad z_3' = 2z_1 + z_2 - 2z_3$.

in general: $\underline{z}' = A(\underline{z})\underline{z}$

§6.1 Vectors

scalar/number

magnitude $z, \pi \in \mathbb{R}$

s, t.

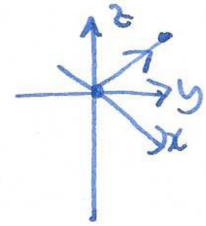
vector

magnitude and direction

2d $\in \mathbb{R}^2$

3d $\in \mathbb{R}^3$

nd $\in \mathbb{R}^n$



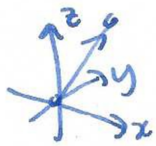
$\underline{v}, \underline{w}$ $\quad \langle x, y, z\rangle \in \mathbb{R}^3$.

scalar multiplication $2\underline{v}$ vector same direction twice as long $\quad \underline{v} = \langle v_1, v_2, v_3\rangle$.

$-\frac{1}{2}\underline{v}$ opposite half as long $\quad 3\underline{v} = \langle 3v_1, 3v_2, 3v_2\rangle$

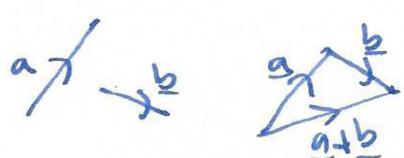
warning $3 + \underline{v}$ doesn't make sense!

length



$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\qquad$ check: $\|\lambda \underline{v}\| = |\lambda|\|\underline{v}\|$.

vector addition

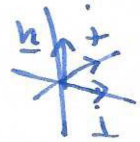


$\underline{a} = \langle a_1, a_2, a_3\rangle$ $\quad \underline{b} = \langle b_1, b_2, b_3\rangle$.

$\underline{a} + \underline{b}$ $\quad \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$.

triangle inequality $\|\underline{a} + \underline{b}\| \leq \|\underline{a}\| + \|\underline{b}\|$

special vectors

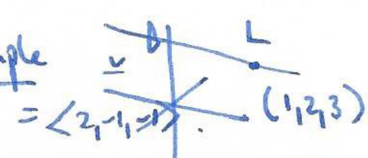


$\underline{v} = \langle v_1, v_2, v_3\rangle = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$

equations of lines

$\underline{r}(t) = \underline{a} + t\underline{v}$

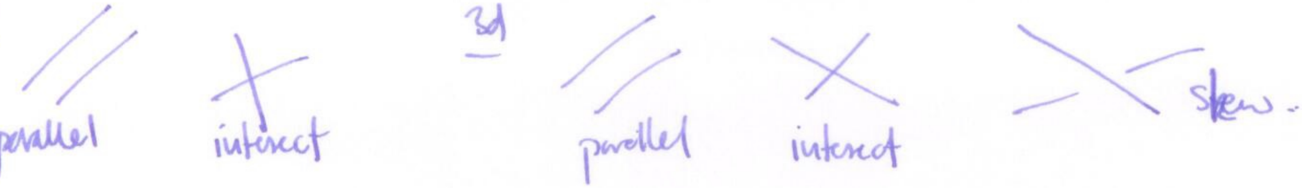
Example



$\underline{r}(t) = \langle 1, 2, 3\rangle + t\langle 2, -1, -1\rangle$

$\underline{s}(t) = \langle 3, 1, 2\rangle + t\langle -4, 2, 2\rangle$

← same!



§6.2 Dot product

Def^n (geometric) $\underline{a}.\underline{b} = \|\underline{a}\|\,\|\underline{b}\|\cos\theta$

(algebraic) $\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$.

so can find θ: $\cos\theta = \dfrac{\underline{a}.\underline{b}}{\|\underline{a}\|\|\underline{b}\|}$

Useful properties

$\underline{a}.\underline{b} = \underline{b}.\underline{a}$

$(\underline{a}+\underline{b}).\underline{c} = \underline{a}.\underline{c} + \underline{b}.\underline{c}$

$\lambda(\underline{a}.\underline{b}) = (\lambda\underline{a}).\underline{b} = \underline{a}.(\lambda\underline{b})$

$\underline{a}.\underline{a} = \|\underline{a}\|^2$

$\underline{a}.\underline{a} = 0 \Rightarrow \underline{a} = \underline{0}$

$\|\alpha\underline{a} + \beta\underline{b}\|^2 = \alpha^2\|a\|^2 + 2\alpha\beta\,\underline{a}.\underline{b} + \beta^2\|\underline{b}\|^2$ (*)

Cauchy-Schwarz

$|\underline{a}.\underline{b}| \le \|\underline{a}\|\|\underline{b}\|$.

warning: 0 number $\underline{0}$ vector $\langle 0,0,0\rangle$

$\underline{a}.\underline{b}.\underline{c}$ doesn't make sense! but $(\underline{a}.\underline{b})\underline{c}$ does.

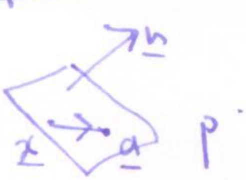
(*) proof: $\|\alpha\underline{a} + \beta\underline{b}\|^2 = (\alpha\underline{a} + \beta\underline{b}).(\alpha\underline{a} + \beta\underline{b})$

$= \alpha^2\underline{a}.\underline{a} + 2\alpha\beta\,\underline{a}.\underline{b} + \beta^2\,\underline{b}.\underline{b}$

$= \alpha^2\|\underline{a}\|^2 + 2\alpha\beta\,\underline{a}.\underline{b} + \beta^2\|b\|^2$. $\square$.

Def^n Two vectors are orthogonal or perpendicular if $\underline{a}.\underline{b} = 0$

Equations of planes



$(\underline{x} - \underline{a}).\underline{n} = 0$

$\underline{n} = \langle a,b,c\rangle$

$\underline{a} = \langle x_0,y_0,z_0\rangle$.

$\underline{x} = \langle x,y,z\rangle$.

$(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle).\langle a,b,c\rangle = 0$

$ax + by + cz = d$.