$\rightsquigarrow \quad Y(t) = c_1 e^{-3t} + c_2 e^{2t}$

$t = \ln(x) \quad \rightsquigarrow \quad y(x) = c_1 e^{-3\ln(x)} + c_2 e^{2\ln(x)} = \frac{c_1}{x^3} + c_2 x^2.$

Alternate method: look for solution
$$y = x^\lambda$$
$$y' = \lambda x^{\lambda-1}$$
$$y'' = \lambda(\lambda-1)x^{\lambda-2}$$

$$x^2 y'' + 2xy' - 6y = 0$$
$$x^2 \lambda(\lambda-1)x^{\lambda-2} + 2x\lambda x^{\lambda-1} - 6x^\lambda = 0$$
$$x^2\left[\lambda^2 - \lambda + 2\lambda - 6\right] = 0$$
$$x^2\left[\lambda^2 + \lambda - 6\right] = 0 \qquad \lambda = -3, 2, \qquad c_1 x^{-3} + c_2 x^2.$$
$$(\lambda+3)(\lambda-2)$$
(check!).

Example $\quad x^2 y'' - 5xy' + 8y = 2\ln(x)$

solve homogeneous problem: $x^2 y'' - 5xy' + 8y = 0$

look for sol$^{\underline{n}}$
$$\left.\begin{aligned} y &= x^\lambda \\ y' &= \lambda x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)x^{\lambda-2}\end{aligned}\right\} \quad x^\lambda\left(\lambda^2 - \lambda - 5\lambda + 8\right) = 0$$
$$(\lambda^2 - 6\lambda + 8)$$
$$(\lambda-4)(\lambda-2) \qquad c_1 x^4 + c_2 x^2$$

undetermined coeffs does not apply!

but does apply to transformed equation: $Y''(t) - 6Y'(t) + 8Y(t) = 2t$
$(x = e^t)$

homogeneous solutions $c_1 e^{4t} + c_2 e^{2t}$.

now look for solution
$$\left.\begin{aligned} Y_p(t) &= At + B \\ Y_p'(t) &= A \end{aligned}\right\} \qquad \begin{aligned} -6A + 8(At+B) &= 2t \\ 8At + (8B - 6A) &= 2t \\ A = 1/4, \quad B &= 3/16 \end{aligned}$$

so general solution $Y(t)$: $\quad c_1 e^{4t} + c_2 e^{2t} + \frac{1}{4}t + \frac{3}{16}$

transform back: $\quad y(x) = c_1 e^{2\ln(x)} + c_2 e^{2\ln(x)} + \frac{1}{4}\ln(x) + \frac{3}{16}$

so $y(x) = c_1 x^4 + c_2 x^2 + \frac{1}{4}\ln(x) + \frac{3}{16}$

Superposition

$$y'' + p(x)y'(x) + q(x)y = f_1(x) + f_2(x) + \cdots + f_N(x) \quad (*)$$

suppose y_i is a solution of $y'' + py' + qy = f_i(x)$

claim: $y_1 + \cdots + y_N$ is a solution of $(*)$.

plug in:
$$(y_1+\cdots+y_N)'' + p(x)(y_1+\cdots+y_N)' + q(x)(y_1+\cdots+y_N)$$
$$= y_1'' + p(x)y_1' + q(x)y_1 + \cdots + y_N'' + p(x)y_N' + q(x)y_N$$
$$= f_1(x) + \cdots + f_N(x) \qquad \square.$$

Example $y'' + 4y = x + 2e^{-2x}$

solve: $y'' + 4y = 0 \qquad y = c_1\cos 2x + c_2 \cancel{\cos}\sin 2x.$

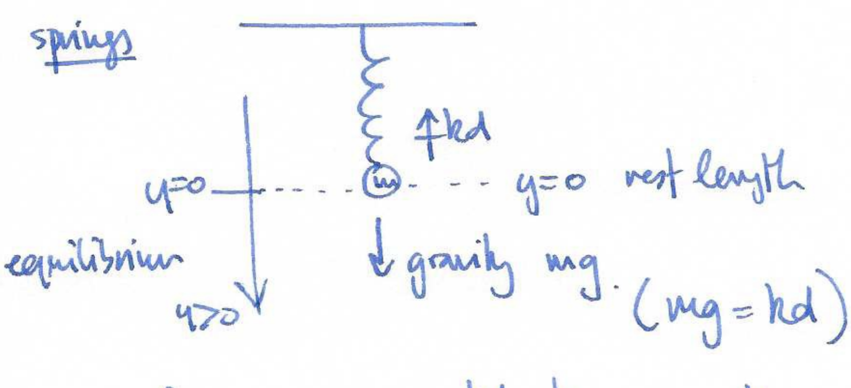
$y'' + 4y = x$ try $y = Ax + B$ get $\frac{1}{4}x$

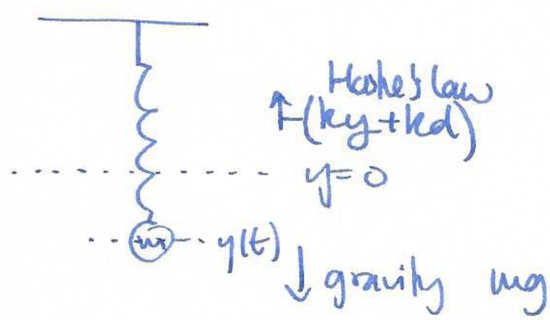
$y'' + 4y = 2e^{-2x}$ try $y = Ae^{-2x}$ get $\frac{1}{4}e^{-2x}$

(or use undetermined coeffs).

Mechanical systems

springs





total force: $\underbrace{mg - kd}_{=0} - ky = -ky$

$F = ma$: $y'' = -ky \qquad y'' + ky = 0 \qquad y = c_1\cos(\sqrt{k}t) + c_2\sin(\sqrt{k}t)$

damping forces / air resistance etc. often proportional to velocity : $-cy'$.

driving force (depends on t) $f(t)$

so $\quad F = -ky - cy' + f(t)$

$$my'' = -ky - cy' + f(t)$$

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = f(t) \qquad \text{(spring equation)}$$

<u>Unforced motion</u> $(f(t) = 0)$

$y'' + \frac{c}{m} y' + \frac{k}{m} y = 0$ look for solution $y = e^{\lambda t}$

$e^{\lambda}\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right) = 0 \qquad \lambda = -\frac{c}{m} \pm \frac{1}{2m}\sqrt{c^2 - 4km}$

<u>case 1</u> $c^2 - 4km > 0$ two distinct real roots

general solution $y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ $\qquad \lambda_1 = -\frac{c}{m} + \frac{1}{2m}\sqrt{c^2 - 4km}$

$\lambda_2 = -\frac{c}{m} - \frac{1}{2m}\sqrt{c^2 - 4km}$

<u>note</u> $\lambda_2 < 0$, $\lambda_1 < 0$ as $\sqrt{c^2 - 4km} < c$.

so $\lim_{t \to \infty} y(t) = 0$. (regardless of initial conditions). (overdamped motion)

<u>case 2</u> $c^2 = 4km$ (critical damping)

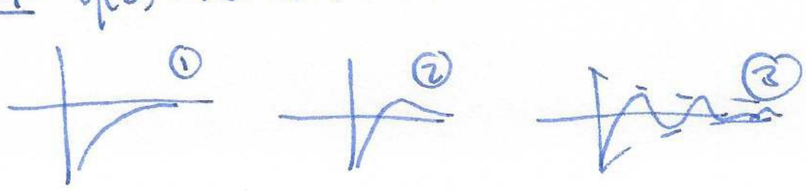
general solution is $y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$, $\lambda = -\frac{c}{m}$

$= (c_1 + c_2 t) e^{-ct/m}$. <u>note</u> $y(t) \to 0$ as $t \to \infty$

<u>case 3</u> $c^2 - 4km$ (underdamped) $\lambda = \alpha \pm i\beta$. $\qquad \alpha = -\frac{c}{m}$.

general solution $e^{\alpha}\left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right)$ $\qquad \beta = \frac{1}{2m}\sqrt{4km - c^2}$

<u>note</u> $y(t) \to 0$ as $t \to \infty$.



Forced motion $\quad y'' + \frac{c}{m}y' + \frac{k}{m}y = f(t)$

Example periodic forcing $f(t) = A\cos(\omega t)$

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{A}{m}\cos(\omega t) \quad (*)$$

already solved homogeneous problem, need to find particular solution

try: $\quad y(t) = a\cos\omega t + b\sin\omega t$

$$y' = -\omega a\sin\omega t + \omega b\cos\omega t$$

$$y'' = -a\omega^2\cos\omega t - b\omega^2\sin\omega t$$

plug in $(*)$:
$$\cos(\omega t)\left[-a\omega^2 + \frac{c}{m}b\omega + \frac{k}{m}a - \frac{A}{m}\right] = 0$$
$$+ \sin(\omega t)\left[-b\omega^2 - \frac{c}{m}a\omega + \frac{k}{m}b\right]$$

both things in brackets must be zero.

$$a\left[-\omega^2 + \frac{k}{m}\right] + b\left[\frac{c\omega}{m}\right] = \frac{A}{m}.$$

$$a\left[-\frac{c\omega}{m}\right] + b\left[-\omega^2 + \frac{k}{m}\right] = 0$$

solution: $\quad a = \dfrac{A(k - m\omega^2)}{(k - m\omega^2)^2 + \omega^2c^2} \qquad b = \dfrac{A\omega c}{(k - m\omega^2)^2 + \omega^2c^2}$

set $\omega_0 = \sqrt{\frac{k}{m}}$ then $\quad y_p(t) = \dfrac{mA(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2c^2}\cos(\omega t)$

(assuming $c \neq 0$, $\omega \neq \omega_0$) $\qquad + \dfrac{A\omega c}{m^2(\omega_0^2 - \omega^2)^2 + \omega^2c^2}\sin(\omega t)$