$$u_1'(x) = -\frac{y_2(x)\, f(x)}{W(x)} \qquad u_2'(x) = \frac{y_1(x)\, f(x)}{W(x)}$$

$$W(x) = y_1 y_2' - y_1' y_2 \neq 0 .$$

so $$u_1(x) = -\int \frac{y_2 f}{W}\, dx \qquad u_2 = \int \frac{y_1 f}{W(x)}\, dx$$

then ~~gen~~ $Y_p = u_1 y_1 + u_2 y_2$ and general solution is

$c_1 y_1 + c_2 y_2 + Y_p$.

Example $y'' - y = x$ (*) solve $y'' - y = 0$ $y_1 = e^x$, $y_2 = e^{-x}$

look for soln $Y_p = u_1 e^x + u_2 e^{-x}$

$$Y_p' = u_1' e^x + u_1 e^x + u_2' e^{-x} - u_2 e^{-x}$$

simplify : $u_1' e^x + u_2' e^{-x} = 0$ (1)

$$Y_p'' = u_1' e^x + u_1 e^x - u_2' e^{-x} + u_2 e^{-x}$$

plug in to (*)

$$u_1' e^x + \cancel{u_1 e^x} - u_2' e^{-x} + \cancel{u_2 e^{-x}} \cancel{- u_1 e^x} \cancel{- u_2 e^{-x}} = x \quad (2)$$

(1) $u_1' e^x + u_2' e^{-x} = 0$

(2) $u_1' e^x - u_2' e^{-x} = x$

(1)-(2) : $2u_2' e^{-x} = x$

$u_2' = \frac{1}{2} x e^x$

$u_2 = \frac{1}{2} x e^x - \frac{1}{2} e^x$

(1)+(2) : $2u_1' e^x = x$

$u_1' = \frac{1}{2} e^{-x} \cdot x$

$u_1 = \frac{1}{2} x e^{-x} + \frac{1}{2} e^{-x}$

$$Y_p = u_1 e^x + u_2 e^{-x} = \left(\tfrac{1}{2}xe^{-x} + \tfrac{1}{2}e^{-x}\right)e^x + \left(\tfrac{1}{2}xe^{x} - \tfrac{1}{2}e^{x}\right)e^{-x}$$

$$= x$$

so $Y = c_1 e^x + c_2 e^{-x} + x$.

§2.3.2 Undetermined coefficients (constant coeffs homogeneous only)

Example ① $y'' - 4y = 8x^2 - 2x$ homogeneous solution $y_1 = e^{2x}$, $y_2 = e^{-2x}$

to find Y_p: try polynomial

$$Y_p = Ax^2 + Bx + C$$
$$Y_p' = 2Ax + B$$
$$Y_p'' = 2A$$

get

$$2A - 4(Ax^2 + Bx + C) = 8x^2 - 2x$$

$$x^2\underbrace{(-4A-8)} + x\underbrace{(-4B+2)} + \underbrace{2A-4C} = 0$$

solve: $A = -2$, $B = 1/2$, $C = -1$.

$Y_p = -2x^2 + \frac{1}{2}x - 1$. general solution $c_1 e^{2x} + c_2 e^{-2x} - 2x^2 + \frac{1}{2}x - 1$.

② $y'' + 2y' - 3y = 4e^{2x}$ homogeneous: $c_1 e^{-3x} + c_2 e^x$

particular: try

$$\left.\begin{aligned} Y_p &= Ae^{2x} \\ Y_p' &= 2Ae^{2x} \\ Y_p'' &= 4Ae^{2x} \end{aligned}\right\}$$

plug in

$$4Ae^{2x} + 4Ae^{2x} - 3Ae^{2x} = 4e^{2x}$$

$$e^{2x}(5A - 4) = 0 \quad A = 4/5.$$

general solution: $c_1 e^{-3x} + c_2 e^x + \frac{4}{5}e^{2x}$.

③ $y'' - 5y' + 6y = -3\sin(2x)$ try: $A\cos(2x) + B\sin(2x)$ $Y_p = \frac{-15}{52}\cos(x) - \frac{3}{52}\sin(x)$.

④ $y'' + 2y' - 3y = 8e^x$ h.s. e^x, e^{-3x} try: $Y_p = Axe^x$ $A = 2$.

⑤ $y'' - 6y' + 9y = 5e^{3x}$ h.s. e^{3x}, xe^{3x} try: $Y_p = Ax^2e^{3x}$ $A = 5/2$

Euler's equation: $y'' + \frac{1}{x} A y' + \frac{1}{x^2} B y = 0$ ⊛ A, B constant

(assume $x>0$)

change of variables $x = e^t$ (or $t = \ln(x)$)

let $Y(t) = y(e^t)$

$$y'(x) = \frac{dY}{dt}\frac{dt}{dx} = Y'(t)\frac{1}{x}$$

so $Y'(t) = x y'(x)$

$$y''(x) = \frac{d}{dx}(y'(x)) = \frac{d}{dx}\left(\frac{1}{x}Y'(t)\right)$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{d}{dx}(Y'(t))$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}\frac{dY'}{dt}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}Y'(t) + \frac{1}{x}Y''(t)\frac{1}{x}$$

$$= -\frac{1}{x^2}(Y''(t) - Y'(t))$$

so $x^2 y''(x) = Y''(t) - Y'(t)$

⊛ : $x^2 y''(x) + x A y'(x) + B y(x) = 0$

then $Y''(t) - Y'(t) + A Y'(t) + B Y(t) = 0$

$$Y'' + (A-1)Y' + BY = 0 \leftarrow \text{2nd order constant coefficient !}$$

Example $x^2 y'' + 2xy' - 6y = 0$ $x = e^t$

$\Leftrightarrow$ $Y'' + Y' - 6Y = 0$

$\rightsquigarrow \quad Y(t) = c_1 e^{-3t} + c_2 e^{2t}$

$t = \ln(x) \quad \rightsquigarrow \quad y(x) = c_1 e^{-3\ln(x)} + c_2 e^{2\ln(x)} = \frac{c_1}{x^3} + c_2 x^2.$

Alternate method: look for solution

$$y = x^\lambda$$
$$y' = \lambda x^{\lambda-1}$$
$$y'' = \lambda(\lambda-1)x^{\lambda-2}$$

$$x^2y'' + 2xy' - 6y = 0$$

$$x^2\lambda(\lambda-1)x^{\lambda-2} + 2x\lambda x^{\lambda-1} - 6x^\lambda = 0$$

$$x^2\left[\lambda^2 - \lambda + 2\lambda - 6\right] = 0$$

$$x^2\left[\lambda^2 + \lambda - 6\right] = 0 \qquad \lambda = -3, 2, \qquad c_1x^{-3} + c_2x^2.$$
$$(\lambda+3)(\lambda-2)$$

(check!).

Example $\quad x^2y'' - 5xy' + 8y = 2\ln(x)$

solve homogeneous problem: $x^2y'' - 5xy' + 8y = 0$

look for sol$^{\underline{n}}$

$$\left.\begin{aligned} y &= x^\lambda \\ y' &= \lambda x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)x^{\lambda-2} \end{aligned}\right\} \quad x^\lambda\left(\lambda^2 - \lambda - 5\lambda + 8\right) = 0$$

$$(\lambda^2 - 6\lambda + 8)$$
$$(\lambda-4)(\lambda-2) \qquad c_1x^4 + c_2x^2$$

undetermined coeffs does not apply!

but does apply to transformed equation: $Y''(t) - 6Y'(t) + 8Y(t) = 2t$
$(x = e^t)$

homogeneous solutions $\quad c_1e^{4t} + c_2e^{2t}.$

now look for solution

$$\left.\begin{aligned} Y_p(t) &= At + B \\ Y'_p(t) &= A \end{aligned}\right\} \qquad \begin{aligned} -6A + 8(At+B) &= 2t \\ 8At + (8B - 6A) &= 2t \\ A = 1/4, \quad B &= 3/16 \end{aligned}$$

so general solution: $\quad Y(t) \quad c_1e^{4t} + c_2e^{2t} + \frac{1}{4}t + \frac{3}{16}$

transform back: $\quad y(x) = c_1e^{2\ln(x)} + c_2e^{2\ln(x)} + \frac{1}{4}\ln(x) + \frac{3}{16}$