<u>Thm 2.1</u> Let p, q, f be cts on an open interval I. Let $x_0 \in I$ and A, B any real numbers. Then $y'' + p(x)y' + q(x)y = f(x)$ with $y(x_0) = A$, $y'(x_0) = B$ has a unique solution defined for all x in I.

<u>Homogeneous equations</u>

<u>Defn</u> A linear 2nd order equation is homogeneous if $f(x) \equiv 0$, i.e.
$y'' + p(x)y' + q(x) = 0$.

<u>Defn</u> If $y_1(x)$ and $y_2(x)$ are functions and c_1, c_2 are numbers, then $c_1 y_1(x) + c_2 y_2(x)$ is a <u>linear combination</u> of y_1 and y_2.

<u>Thm</u> ~~Eq~~: linear combinations of solutions to a homogeneous equation are homogeneous.

<u>Proof</u> suppose $y'' + p(x)y' + q(x) = 0$ has solutions $y_1(x)$ and $y_2(x)$

check $c_1 y_1 + c_2 y_2$ is a solution: $(c_1 y_1(x) + c_2 y_2(x))'' + p(x)(c_1 y_1(x) + c_2 y_2(x))' + q(x)$

$$= c_1 y_1'' + c_2 y_2'' + p(x) c_1 y_1' + p(x) c_2 y_2' + q(x)$$

$$= c_1 \underbrace{(y_1'' + p y_1' + q)}_{=0} + c_2 \underbrace{(y_2'' + p y_2' + q)}_{=0} = 0 \quad \square.$$

<u>Note</u> $c_1 = c_2 = 0 \Rightarrow y = 0$ is a solution!

<u>Defn</u> two functions f and g are linearly dependent on an open interval $I = (a, b)$ if either $f(x) = c g(x)$ or $g(x) = c f(x)$ for all $x \in I$, c constant.

If f and g are not linearly dependent, we say they are <u>linearly independent</u>

<u>Example</u> $y_1(x) = \sin(x)$ $\quad y_2(x) = \cos(x)$ linearly indep.

check: $y_1(0) = 0 \quad y_2(0) = 1$. $\left(\Rightarrow y_1(x) = 0 \cdot y_2(x)\right)$.

but $y_1(\frac{\pi}{2}) = 1 \quad y_2(\frac{\pi}{2}) = 0$ #.

Example $y''+y=0$ has solutions $y_1(x)=\sin(x)$, $y_2(x)=\cos(x)$, (18)

linearly indep, so general solution is $c_1y_1(x)+c_2y_2(x) = c_1\sin(x)+c_2\cos(x)$.

Test for independence: Def^n The Wronskian $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$

$$\left(W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \right)$$

Th^m Let y_1 and y_2 be solutions of $y''+p(x)y'+q(x)y=0$ an open interval I

Then ① either $W(x)\equiv 0$ for all x in I or $W(x)\neq 0$ for all x in I.

② y_1 and y_2 are linearly independent iff $W(x)\neq 0$ on I.

Example $y''+y=0$ has solutions $y_1(x)=\cos(x)$
$y_2(x)=\sin(x)$

then $W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$.

Example $y''+xy=0$ has solutions $y_1(x) = 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \dots$

$y_2(x) = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \dots$

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

Th^m Let y_1 and y_2 be linearly independent solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Then every solution of this d.e. is of the form $c_1y_1+c_2y_2$, i.e. a linear combination of y_1 and y_2.

Def^n y_1, y_2 solutions of $y''+p(x)y'+q(x)y'=0$ on I

① y_1, y_2 form a fundamental set of solutions iff they are linearly indep.

② in this case, the general solution is $c_1y_1+c_2y_2$.

<u>Proof</u> Let ϕ be a solution of $y''+py'+qy=0$ on I

aim: show $\phi = c_1y_1+c_2y_2$.

pick $x_0\in I$. Let $\phi(x_0)=A$, $\phi'(x_0)=B$, then ϕ is the unique solution of the initial value problem: $y''+py'+qy=0$, $y(x_0)=A$, $y'(x_0)=B$.

consider ① $c_1y_1(x_0)+c_2y_2(x_0)=A$

② $c_1y_1'(x_0)+c_2y_2'(x_0)=B$ } solve for c_1,c_2

$y_2'(x_0)$① $-y_2(x_0)$②: $c_1\underbrace{\left(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\right)}_{W(x_0)\neq 0!}=Ay_2'(x_0)-By_2(x_0)$

$$c_1=\frac{Ay_2'(x_0)-By_2(x_0)}{W(x_0)}$$

$y_1'(x_0)$① $-y_1(x_0)$②: $c_2\underbrace{\left(y_1'(x_0)y_2(x_0)-y_1(x_0)y_2'(x_0)\right)}_{-W(x_0)\neq 0!}=Ay_1'(x_0)-By_1(x_0)$

$$c_2=\frac{By_1(x_0)-Ay_1'(x_0)}{W(x_0)}$$

so $c_1y_1(x)+c_2y(x)$ satisfies IVP, uniqueness $\Rightarrow \phi(x)=c_1y_1+c_2y_2$ □.

<u>Example</u> solve $y''-3y'+2y=0$, $y(0)=1$, $y'(0)=4$.

Look for sol^n $y(x)=e^{\lambda x}$: $y'=\lambda e^{\lambda x}$, $y''=\lambda^2e^{\lambda x}$.

$$\lambda^2e^{\lambda x}-3\lambda e^{\lambda x}+2e^{\lambda x}=0$$

$$e^{\lambda x}\left[\lambda^2-3\lambda+2\right]=0 \qquad (\lambda-2)(\lambda-1)=0$$

solutions: $y=e^{2x}, e^{x}$, general solution: $y(x)=c_1e^{x}+c_2e^{2x}$

$y(0)=c_1+c_2=1$ } $c_2=3$

$y'(0)=c_1+2c_2=4$ } $c_1=-2$

Non-homogeneous case

Example $y''+4y = 8x$

Facts : $y''+4y=0$ has general solution $y = c_1\cos 2x + c_2 \sin 2x$

$y''+4y=8x$ has a particular solution $y=2x$

claim the general solution is $\underbrace{c_1\cos 2x + c_2\sin 2x}_{\text{homogeneous soln}} + \underbrace{2x}_{\text{particular solution}}$

Thm 2.5 Let Y_p be a solution of $y''+p(x)y'+q(x)y = f(x)$ ⊛ and let y_1, y_2 be lin indep solutions of the homogeneous equation $y''+p(x)y'+q(x)y=0$

Then every solution of ⊛ is of the form $c_1y_1+c_2y_2+Y_p$.

Method solve $y''+py'+qy=f(x)$

1. solve homogeneous problem $y''+py'+qy=0$
2. find any solution Y_p
3. general solution is $c_1y_1+c_2y_2+Y_p$.

§2.2 Constant coefficients.

$y''+ay'+by=0$ a,b numbers.

Look for solution $y=e^{\lambda x}$: $y'=\lambda e^{\lambda x}$ $y''=\lambda^2 e^{\lambda x}$

$\lambda^2 e^{\lambda x}+a\lambda e^{\lambda x}+be^{\lambda x}=0$

$e^{\lambda x}(\lambda^2+a\lambda+b)=0$ $\lambda = \dfrac{-a\pm\sqrt{a^2-4b}}{2}$.

1. distinct real roots $\lambda_1\ \lambda_2$
2. repeated roots λ_1, λ_1
3. complex roots

① distinct real roots: general solution is $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example $y''-y'-6y=0$ $\quad \lambda^2-\lambda-6 = (\lambda-3)(\lambda+2)$ $\quad y=c_1e^{3x}+c_2e^{-2x}$.

② ~~repeated~~ complex roots $\lambda = \alpha \pm \beta i$

Example $y''\cancel{+4y'}+4y=0$ $\quad \lambda^2\cancel{+4\lambda}+4=$ $\quad \lambda = \pm 2i$

$c_1e^{2ix}+c_2e^{-2ix}$? recall $\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\}$

$\updownarrow$

$c_1(\cos 2x + i\sin 2x) + c_2(\cos 2x - i\sin 2x)$ $\quad e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$(c_1+c_2)\cos 2x + i(c_1-c_2)\sin 2x$ $\quad e^{i\theta}-e^{-i\theta} = 2\sin\theta$

$\left.\begin{aligned} \hookrightarrow c_1=c_2 &: \cos 2x \\ c_1=-c_2 &: \sin 2x \end{aligned}\right\}$ general solution $c_1\cos 2x + c_2\sin 2x$.

In general $\lambda = \alpha \pm \beta i$ get general solution $c_1 e^{\alpha}\cos\beta x + c_2 e^{\alpha}\sin\beta x$

Example $y''+2y+2y=0$ $\quad \lambda = 1\pm i$.

③ repeated roots: Example $y''+4y'+4y=0$ (†) $\quad \lambda^2+4y+4=0$

$(\lambda+2)^2=0 \quad \lambda=-2,-2$.

problem: e^{-2x}, e^{-2x} not independent!

find second solution: (reduction of order) look for a solution

$y(x) = u(x)e^{-2x}$.

$y'(x) = u'e^{-2x} + u\cdot -2e^{-2x}$

$y''(x) = u''e^{-2x} + u'\cdot -2e^{-2x} + u'\cdot -2e^{-2x} + u\cdot 4e^{-2x}$

$= u''e^{-2x} - 4u'e^{-2x} + u4e^{-2x}$.

subs in (†): $\left.\begin{aligned} & u''e^{-2x} - 4u'e^{-2x} + u4e^{-2x} \\ & +4u'e^{-2x} - 8ue^{-2x} \\ & +4ue^{-2x} \end{aligned}\right\}$ $\quad u''(x)\underbrace{e^{-2x}}_{\neq 0} = 0$

$\Rightarrow u''(x) = 0$

so $u(x) = cx+d$, so $(cx+d)e^{-2x}$ is a solution

so general sol^n is $c_1 e^{-2x} + c_2 x e^{-2x}$.

Fact if $y'' + ay' + by = 0$ has repeated root λ, then general sol^n is $c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

§2.3.1 Variation of parameters.

$$y'' + p(x)y' + q(x)y = f(x) \quad (*)$$

suppose we have solutions y_1 and y_2 for homogeneous equation.

Q: how do we find particular solution Y_p?

Try: Look for solution $Y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

then $Y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$

simplifying assumption: $u_1'y_1 + u_2'y_2 = 0 \quad (1)$

$$Y_p'(x) = u_1y_1' + u_2y_2'$$

$$Y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

plug in to $(*)$:

$$u_1'y_1' + \underbrace{u_1y_1'' + pu_1y_1' + qu_1y_1}_{=0} + u_2'y_2' + \underbrace{u_2y_2'' + pu_2y_2' + qu_2y_2}_{=0} = f(x) \quad (2)$$

$$\left.\begin{aligned} (1)\quad & u_1'y_1 + u_2'y_2 = 0 \\ (2)\quad & u_1'y_1' + u_2'y_2' = f(x) \end{aligned}\right\} \text{solve}$$