Def$^{\underline{n}}$ A <u>Bernoulli equation</u> is one of the form $y' + P(x)y = R(x)y^{\alpha}$ (α const)

(linear if $\alpha = 0$ and separable if $\alpha = 1$)

if $\alpha \neq 1$ becomes linear w/ change of variable $v = y^{1-\alpha}$

<u>Example</u> $y' + \frac{2}{x}y = \frac{3}{x}y^2$ ($\alpha = 2$ try $v = y^{-1}$).

$y = \frac{1}{v}$, $y' = -\frac{1}{v^2}v'$

$$-\frac{1}{v^2}v' + \frac{2}{x}\frac{1}{v} = \frac{3}{x}\frac{1}{v^2}$$

$$-v' + \frac{2}{x}v = \frac{3}{x}$$ linear in v; integrating factor

$$v' - \frac{2}{x}v = -\frac{3}{x} \qquad g(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = x^{-2}.$$

$$x^{-2}v' - 2x^{-3}v = -3x^{-3}$$

check! $$(vx^{-2})' = -3x^{-3}$$

$$\frac{v}{x^2} = +\frac{3}{2}x^{-2} + c$$

$$v = \frac{3}{2} + cx^2$$

$$\frac{1}{y} = \frac{3}{2} + cx^2$$

$$y = \frac{1}{\frac{3}{2} + cx^2}.$$

Def$^{\underline{n}}$ A differential equation of the form $y' = P(x)y^2 + Q(x)y + R(x)$ 12

is called a <u>Ricatti equation</u> 15

Sol$^{\underline{n}}$: if you can find a particular solution $s(x)$, then change of var $y = s(x) + \frac{1}{z}$ gives a linear equation in x and z.

Example

$$y' = \frac{1}{x}y^2 + \frac{1}{x}y - \frac{2}{x}$$

check: $y = 1$ is a sol$^{\underline{n}}$. $s(x) = 1$.

set $y = 1 + \frac{1}{z}$ so $y' = -\frac{1}{z^2}\cdot z'$

$$\rightsquigarrow \quad -\frac{1}{z^2}z' = \frac{1}{x}\left(1+\frac{1}{z}\right)^2 + \frac{1}{x}\left(1+\frac{1}{z}\right) - \frac{2}{x}$$

$$\rightsquigarrow \quad z' + \frac{3}{x}z = -\frac{1}{x}$$

integrating factor x^3.

$$\rightsquigarrow \quad x^3 z = -\frac{1}{3}x^3 + c$$

$$z = -\frac{1}{3} + \frac{c}{x^3}.$$

$$y = 1 + \frac{1}{z} = 1 + \frac{1}{-1/3 + c/x^3} = \frac{k + 2x^3}{k - x^3} \quad (k = 3c)$$

Example Electrical circuits.

RLC circuits: R resistor, L inductor, C capacitor

$E(t)$ electromotive force.

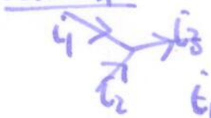
$i(t)$ current, $q(t)$ charge: $i(t) = q'(t)$

voltage drops

iR $\quad q/C \quad Li'$

Kirchhoff's laws

current

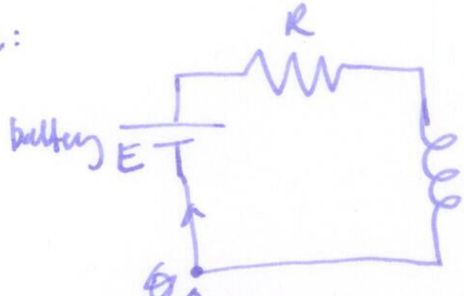


$i_1 + i_2 + i_3 = 0$

voltage

voltage change is zero around closed loop.

simple example:



$$+E - iR - Li' = 0$$

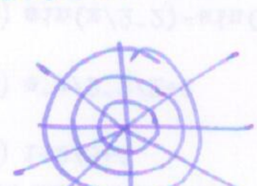
linear differential equation: $\quad i' + \frac{E}{R} i = \frac{E}{L}$

general solution: $\quad i(t) = \frac{E}{R} + k e^{-Rt/L}$ $\qquad \lim_{t \to \infty} i(t) = \frac{E}{R}.$

Orthogonal trajectories

two intersecting curves are orthogonal if their tangents are perpendicular.

e.g.



$x^2 + y^2 = k^2$

$\cancel{xy = c}$ $\quad x = cy$

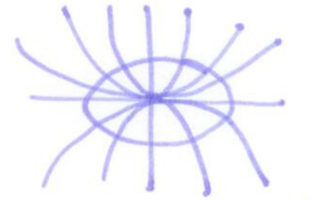
given some curves F, how do we find orthogonal curves?

assume curves F come from graphs of an equation $F(x,y,k) = 0$ (k corresponds to different curves) $\rightsquigarrow$ get $y' = f(x,y)$ (a differential equation which has solutions F. Then orthogonal curves satisfy

$y' = \dfrac{-1}{f(x,y)}$ $\leftarrow$ solve this.

Example $F =$ $y = kx^2$, parabolas through $(0,0)$

$y' = 2kx$ is differential equation which has F as solutions



orthogonal curves: $\cancel{y' = \frac{1}{2kx}}$ warning k not constant on orthogonal curves!

need k in terms of x and y: $k = \frac{y}{x^2}$.

$$y' = -\frac{1}{2kx} = -\frac{x^2}{2yx} = -\frac{x}{2y} \Big\} \text{ separable}$$

$$\frac{dy}{dx} = -\frac{x}{2y} \qquad \int 2y\,dy = \int -x\,dx \qquad y^2 = -\frac{1}{2}x^2 + c$$

$\boxed{x^2 + 2y^2 = c.}$ ellipses.

Existence and uniqueness

Warning: solutions do not always exist; are not always unique.

Example • $y' = 2\sqrt{y}$, $y(0) = -1$ general solution $y = (x+c)^2$
$y(0) \neq -1$.

• $y' = 2\sqrt{y}$, $y(1) = 0$ has solution: $y(x) = 0$

also has solution: $y(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)^2 & x \geq 1 \end{cases}$

Thm Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous for all (x,y) in a rectangle R centered at (x_0, y_0). Then there is a positive number h such that the initial value problem $y' = f(x,y)$; $y(x_0) = y_0$ has a unique solution defined for at least $x_0 - h < x < x_0 + h$

§2.1 Second order differential equations

<u>Defn</u> A second order differential equation has the form $F(x,y,y',y'')=0$

<u>Examples</u> $y''+y'+y=\sin(x)$ $\quad y''=x^3$ $\quad xy''-\cos(y)=e^x$

A solution is a function which satisfies the equation.

<u>Example</u> $y''+16y=0$ has solution $\phi(x)=6\cos(4x)-17\sin(4x)$ check!

<u>Defn</u> A <u>linear</u> second order differential equation has the form

$$R(x)\,y''+P(x)\,y'+Q(x)\,y=S(x) \qquad (R,P,Q,R,S \text{ cts on some interval})$$

if $R(x)\neq 0$ can divide to get special linear differential equation

$$y''+p(x)\,y'+q(x)\,y=f(x).$$

<u>Q</u>: do solution exist? are they unique? how can we find them?

§2.2 Solution of linear equations

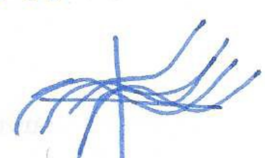
<u>Example</u> $\quad y''-12x=0$

$$y''=12x$$

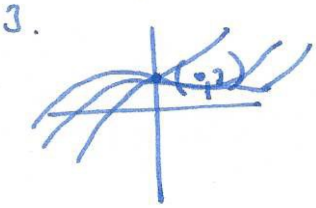
integrate: $\int y''\,dx = y' = \int 12x\,dx = 6x^2+c$

integrate again: $\int y'\,dx = y = \int 6x^2+c\,dx = 2x^3+cx+d$

solutions with <u>2</u> parameters; many solutions through each point:



particular solution: suppose $y(0)=3$: $\quad 3=2\cdot 0+c\cdot 0+d \Rightarrow d=3.$



still many solutions $y=2x^3+cx+3$

suppose $y'(0)=-1$: $\quad y'=6x^2+c$

$$-1=6\cdot 0+c \Rightarrow c=-1.$$

now unique solution: $y=2x^3-x+3$

<u>Facts</u> 2nd order: general solution has <u>2</u> parameters, need <u>2</u> initial conditions