Example $y' + y = x$ $\qquad$ $y' + p(x)\,y = q(x)$

$$p(x) = 1 \qquad q(x) = x$$

int. factor $g(x) = \oint e^{\int p(x)dx} = e^{\int 1 dx} = e^{x}$

$$e^{x}y' + e^{x}y = xe^{x}$$

parts $\int u'v\,dx = uv - \int uv'\,dx$

$$(e^{x}y)' = xe^{x}$$

$$ye^{x} = \int xe^{x}dx = xe^{x} - e^{x} + c$$

$$y = x - 1 + ce^{-x}.$$ (general solution)

Example $y' + \frac{1}{1+x}\,y = x$ $\qquad$ $e^{\int \frac{1}{1+x}dx} = e^{\ln|1+x|} = |1+x|.$

$1+x > 0$:

$$(1+x)y' + y = x(1+x) = x^2 + x$$

$$((1+x)y)' = x^2 + x$$

$$(1+x)\,y = \tfrac{1}{3}x^3 + x^2 + c \qquad y = \frac{\frac{1}{3}x^3 + \frac{1}{2}x^2 + c}{1+x}.$$

$1+x < 0$: etc.

§1.3 Exact equations

recall $\phi(x,y)$. $\qquad$ differential: $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$

$$\frac{\partial \phi}{\partial x} \qquad \frac{\partial \phi}{\partial y}$$

intuition: $\Delta\phi \approx \frac{\partial \phi}{\partial x}\Delta x + \frac{\partial \phi}{\partial y}\Delta y$

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

$d\phi = 0 \Rightarrow \phi(x,y) = \text{constant}.$

if ϕ "nice"
cts, differentiable
etc.

§1.4 Exact Differential equations

given $y' = f(x,y)$ we can write this as $M(x,y) + N(x,y)y' = 0$

suppose there is a function $\phi(x,y)$ s.t. $\underbrace{M = \frac{\partial \phi}{\partial x} \text{ and } N = \frac{\partial \phi}{\partial y}}$ (*)

if so, we say diff equation is <u>exact</u>

then : $\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$

chain rule: $\frac{d}{dx}\phi(x, y(x)) = 0$

Def ϕ is called the <u>potential function</u> for the exact d.e.

$\Rightarrow \quad \phi(x, y(x)) = C$ constant

gives <u>implicit</u> solutions to differential equation (if we can solve for y, becomes explicit).

<u>Example</u> $\frac{dy}{dx} = -\frac{y+\cos(x+y)}{x+\cancel{\sin}\cos(x+y)} \Leftrightarrow y + \cos(x+y) + (x + \cancel{\sin}\cos(x+y))y' = 0$

<u>claim</u> exact, with $\phi(x,y) = xy + \sin(x+y)$, check:

$\frac{\partial \phi}{\partial x} = y + \cos(x+y) \quad \frac{\partial \phi}{\partial y} = x + \cos(x+y)$ ✓.

so (implicit) solutions are $\phi(x,y) = C \quad xy + \cos(x+y) = C$
(can't solve for y).

<u>Q:</u> how can we tell if a differential equation is exact?

recall.

$$\begin{tikzcd}[column sep=small] & \phi(x,y) \arrow[dl] \arrow[dr] & \\ \frac{\partial \phi}{\partial x} \arrow[d] & & \frac{\partial \phi}{\partial y} \arrow[dll] \\ \frac{\partial^2 \phi}{\partial y \partial x} \quad \frac{\partial^2 \phi}{\partial x \partial y} & & \end{tikzcd}$$

if ϕ is "nice" (differentiable w/ cts derivatives) then $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$.

so $M + Ny' = 0$ is exact if $M = \frac{\partial \phi}{\partial x}$ $N = \frac{\partial \phi}{\partial y}$

$\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example $y + y' = 0$

$M = y$ $N = 1$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial N}{\partial x} = 0$ $1 \neq 0$ not exact.

§1.5 Integrating factors for exact equations

Example $x - xy - y' = 0$ not exact. $M = x - xy$ $\frac{\partial M}{\partial y} = -x$

$N = -1$ $\frac{\partial N}{\partial x} = 0$. $\neq$

Q: can we make it exact by multiplying by a function $\mu(x,y)$?

(called an integrating factor if you can find one)

$\mu(x - xy) - \mu y' = 0$ $M = \mu(x - xy)$ $\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}(x - xy) + -\mu x$

$N = -\mu$ $\frac{\partial N}{\partial x} = -\frac{\partial \mu}{\partial x}$

want: $-\frac{\partial \mu}{\partial x} = (x - xy)\frac{\partial \mu}{\partial y} - x\mu$

(partial differential equation, much harder) but look for special solution $\mu(x)$.

$\rightsquigarrow + \frac{d\mu}{dx} = + x\mu$

$\int \frac{d\mu}{\mu} = \int x \, dx$

$\ln|\mu| = \frac{1}{2}x^2 + c$ (choose $c = 0$)

$\mu = e^{\frac{1}{2}x^2}$

gives: $(x - xy)e^{\frac{1}{2}x^2} - e^{x^2/2} y' = 0$

this is exact, with

$\phi(x,y) = (1 - y)e^{x^2/2} = c$

i.e. $y = 1 - ce^{-x^2/2}$

§1.6 Homogeneous equations

Defn A homogeneous equation has the form $y' = f\left(\frac{y}{x}\right)$

Example $y' = \dfrac{\cancel{x}y}{x+y} = \dfrac{y/x}{1+y/x}$ Non-example: $y' = x^2 y$.

Solution: set $y = ux$, then homogeneous becomes separable.

$y' = f\left(\frac{y}{x}\right)$ $\quad y = ux \iff u = y/x$

$y' = u'x + u$

$\downarrow$

$u'x + u = f(u)$

$u'x = f(u) - u$

$\dfrac{1}{f(u)-u}\dfrac{du}{dx} = \dfrac{1}{x}$

$\displaystyle\int \frac{1}{f(u)-u}\,du = \int \frac{1}{x}\,dx$

etc.

Example $xy' = \dfrac{y^2}{x} + y$

$y' = \left(\dfrac{y}{x}\right)^2 + \dfrac{y}{x}$

$y = ux$

$y' = u'x + u$ $\quad u'x + u = u^2 + u$

$u'x = u^2$

$\displaystyle\int \frac{1}{u^2}\,du = \int \frac{1}{x}\,dx$

$-\dfrac{1}{u} = \ln|x| + c$

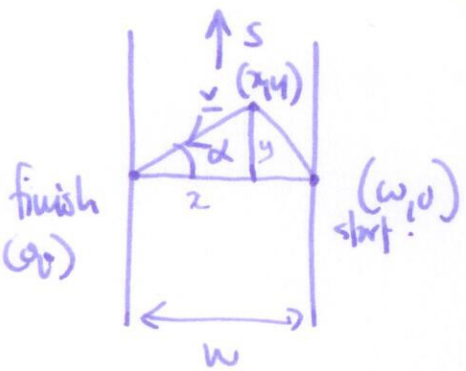
$u = \dfrac{-1}{\ln|x| + c}$

$\dfrac{y}{x} = \dfrac{-1}{\ln|x| + c}$

$y = \dfrac{-x}{\ln|x| + c}$

Pursuit problem

swim across river, always heading to same point



$$\frac{dx}{dt} = -v\cos\alpha$$

$$\frac{dy}{dt} = s - v\sin\alpha$$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{s - v\sin\alpha}{-v\cos\alpha} = \tan\alpha - \frac{s}{v}\sec\alpha$$

$\tan\alpha = \dfrac{y}{x}$, $\sec\alpha = \dfrac{\sqrt{x^2+y^2}}{x}$

$$\frac{dy}{dx} = \frac{y}{x} - \frac{s}{v}\frac{\sqrt{x^2+y^2}}{x} = \frac{y}{x} - \frac{s}{v}\sqrt{1+\left(\frac{y}{x}\right)^2} \qquad (\text{homogeneous!})$$

$y = ux$

$y' = u'x + u$

$$u'x + u = u - \frac{s}{v}\sqrt{1+u^2}$$

$$\int\frac{du}{\sqrt{1+u^2}} = \int -\frac{s}{v}\frac{1}{x}\,dx$$

$$\ln\left|u + \sqrt{1+u^2}\right| = -\frac{s}{v}\ln|x| + c$$

$$\left|u + \sqrt{1+u^2}\right| = e^c e^{-s\ln|x|/v}$$

$$u + \sqrt{1+u^2} = K x^{-s/v}$$

solve for u:

$$u = \frac{1}{2}Kx^{-s/v} - \frac{1}{2}Kx^{s/v}$$

$u = \dfrac{y}{x}$

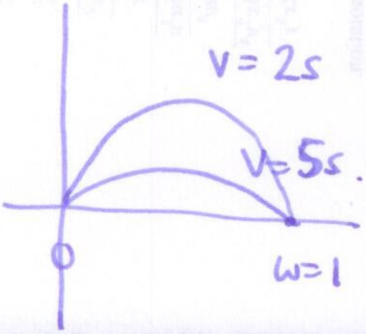
$$y = \frac{1}{2}Kx^{1-s/v} - \frac{1}{2}Kx^{1+s/v}$$

find K: $y(w) = 0$: $\frac{1}{2}Kw^{1-s/v} = \frac{1}{2}Kw^{1+s/v} \Rightarrow K = w^{s/v}$

11

final sol$^{\underline{n}}$

$$y(x) = \frac{w}{2}\left[\left(\frac{x}{w}\right)^{1-s/v} - \left(\frac{x}{w}\right)^{1+s/v}\right].$$



Q: what about $v = s$

$v = \frac{1}{2}s$?