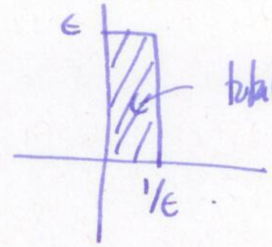
§3.5 Impulses and the delta function



total area 1 (unit impulse)

intuition : consider $\delta_\epsilon(t) = \frac{1}{\epsilon}\left(H(t) - H(t-\epsilon)\right)$

would like : $\delta(t) = \lim_{\epsilon \to 0+} \delta_\epsilon(t)$

$\delta(t)$ not a function. It is a distribution or linear functional : $\{\text{functions}\} \to \mathbb{R}$

$$\delta(f) = \int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$$

Laplace transform : $\delta_\epsilon(t-a) = \frac{1}{\epsilon}\left[H(t-a) - H(t-a-\epsilon)\right]$

$$L(\delta_\epsilon(t-a)) = \frac{1}{\epsilon}\left[\frac{1}{s}e^{-as} - \frac{1}{s}e^{-(a+\epsilon)s}\right] = \frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s}.$$

$$L(\delta(t-a)) = \lim_{\epsilon \to 0+} \frac{e^{-as}(1-e^{-\epsilon s})}{\epsilon s} = e^{-as} \lim_{\epsilon \to 0} \frac{se^{-\epsilon s}}{s} = e^{-as}$$

$(a=0 : \quad L(\delta(t)) = 1)$.

Thm $\displaystyle\int_0^\infty f(t)\delta(t-a)\,dt = f(a)$ $\square$.

Note $\cancel{\int_0^\infty f(t)\delta(t-a)\,dt = f(a)} \Leftrightarrow$ convolutions:

$$f * \delta = \int_0^t f(t-u)\delta(u)\,du = f(t) \qquad \text{so} \qquad f*\delta = f$$

Example solve $y'' + 2y' + 2y = \delta(t-3)$ $\quad y(0)=0, y'(0)=0$

$$L(y'') + 2L(y') + 2L(y) = L(\delta(t-3)).$$

$$s^2Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0} + 2(sY - \underbrace{y(0)}_{0}) + 2Y = e^{-3t}$$

$$Y(s^2+2s+2) = e^{-3t}$$

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$$Y(s) = \frac{1}{s^2+2s+2}e^{-3s} = \frac{1}{(s+1)^2+1}e^{-3s}$$

$$L^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}\sin(t)$$

so $$y(t) = H(t-3)e^{-(t-3)}\sin(t-3)$$