MTH 330 Differential Equations (Applied Mathematical Analysis I) ①

Joseph Maher joseph.maher@csi.cuny.edu

web page : www.math.csi.cuny.edu/~maher

office : 1S-222 office hours M 4:40-6:20
W 4:40-5:30

- math tutoring 1S-214
- students with disabilities

Text: Advanced Engineering Mathematics, O'Neil

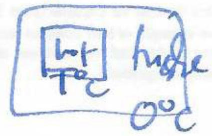
HW: webworks / quizzes

§1.1 Introduction

equation: $x^2+2x+1=0$ solution: number x

differential equation: $\frac{d^2y}{dx^2}+\frac{dy}{dx}+1=0$ solution: function $y(x)$

useful in practice: Newton's law of cooling: temp $T(t)$ rate of cooling proportional to temperature difference:



$\frac{dT}{dt}=\lambda T$.

checking solutions

If we have a solution we can check it works:

$x^2+2x+1=0$, try: $x=-1$ $(-1)^2+2(-1)+1=1-2+1=0$ ✓.

$\frac{dT}{dt}=\lambda T$, try $T=e^{\lambda t}$, $\frac{dT}{dt}=\lambda e^{\lambda t}$ $\lambda e^{\lambda t}=\lambda e^{\lambda t}$ ✓.

simple examples:

Defn A differential equation is separable if $\frac{dy}{dx}=F(x)G(y)$

Example $\frac{dy}{dx}=xy$ non-example $\frac{dy}{dx}=x+y$

solving separable equations:

$$\frac{dy}{dx} = F(x)\, G(y)$$

$$\frac{1}{G(y)} \frac{dy}{dx} = F(x) \qquad (G(y) \neq 0).$$

integrate wrt x: $\displaystyle\int \frac{1}{G(y)} \frac{dy}{dx}\, dx = \int F(x)\, dx$

chain rule: $\displaystyle\int \frac{1}{G(y)}\, dy = \int F(x)\, dx$

integrate, get $g(y) = f(x) + c$; try and solve for $y = h(x)$

notation $\dfrac{dy}{dx} = F(x)\, G(y)$

$$\int \frac{1}{G(y)}\, dy = \int F(x)\, dx \quad \ldots$$

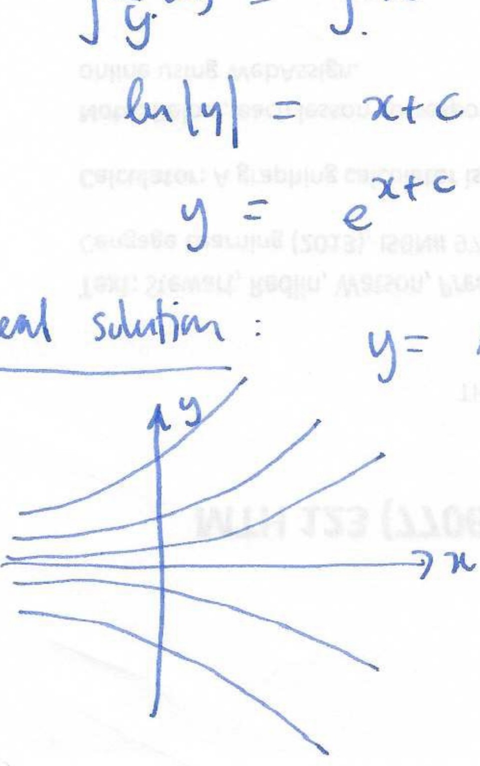
Example $\dfrac{dy}{dx} = y$

$$\int \frac{1}{y}\, dy = \int dx \qquad (y \neq 0)$$

$$\ln|y| = x + c$$

$$y = e^{x+c} = e^x \cdot e^c = A e^x$$

general solution: $y = Ae^x$ and $y(x) = 0$.



particular solution, eg. $A = 4$: $y = 4e^x$.

initial value problem: $y' = 4y$ and $y(0) = 2$ ③

find general solution: $y = \begin{cases} Ae^{x} \\ 0 \end{cases}$ set $x=0$: $2 = Ae^{0} \Rightarrow A = 2$

soln $y(x) = 2e^{x}$.

Example $x^3 y' = 1+y$

$$\int \frac{1}{1+y} \frac{dy}{dx} = \frac{1}{x^3} \quad (1+y \neq 0, \ x \neq 0)$$

$$\int \frac{1}{1+y} dy = \int \frac{1}{x^3} dx$$

$$\ln|1+y| = -\tfrac{1}{2}x^{-2} + c$$

$$1+y = e^{-\frac{1}{2x^2}+c} = Ae^{-1/2x^2} \quad (A \neq 0).$$

$$y(x) = \begin{cases} -1 + Ae^{-\frac{1}{2x^2}} & (A \neq 0) \\ y = -1 \end{cases}$$

initial value problem $y(1) = 1$ (note $x \neq 0$!).

$$1 = -1 + Ae^{-\frac{1}{2}}$$

$2 = Ae^{-1/2}$ $A = 2e^{1/2}$ so $y(x) = -1 + 2e^{1/2}.e^{-1/2x^2}$.

Example: Newton's law of cooling

rate of cooling $\propto$ temperature difference

$$\frac{dT}{dt} = k(T-A)$$

suppose you find a dead body at 12noon, in a room at 70°F, with temperature 80°F. How long have they been dead? Assume normal body temperature is 98°F.

$$\frac{dT}{dt} = k(T-70)$$

$$\int \frac{dT}{T-70} = \int k\,dt$$

$$\ln|T-70| = kt + c$$

$$T-70 = e^{kt+c}$$

$$T = 70 + Ae^{kt}$$

wait half an hour, measure temp again: $t=30$, $T=75$.

$t=0$: $\quad 80 = 70 + Ae^{0} \quad \Rightarrow \quad A = 10$

$t=30$: $\quad 75 = 70 + 10e^{k30} \quad \Rightarrow \quad 5 = 10e^{30k}$

$$\frac{1}{2} = e^{30k}$$

$$\ln(1/2) = 30k$$

$$k = \frac{1}{30}\ln(1/2) \approx -\frac{1}{100}.$$

$$T(t) = 70 + 10e^{-t/100}$$

want t when $T = 98$

$$98 = 70 + 10e^{-t/100}$$

$$28 = 10e^{-t/100}$$

$$2.8 = e^{-t/100}$$

$$\ln(2.8) = -t/100$$

$$t = -100\ln(2.8) \approx -44.7.$$

approx 45 mins.

§1.2 Linear equations

Defn A first order differential equation is linear if it has the form

$$y' + p(x)\,y = q(x)$$

p, q functions of x.

Non-linear: $y' = y^2$ etc. linear: $y' + x^2 y = x^3$.

Strategy: • product rule: $(ab)' = a'b + ab'$

$$(y \cdot a)' = y'a + ya'$$

aim: multiply by function so we have a ~~reverse~~ product rule.

• note: $(e^{f(x)})' = e^{f(x)} \cdot f'(x)$. want $f'(x) = p(x)$.

$$f(x) = \int p(x)\,dx.$$

Defn the integrating factor for $y' + py = q$ is $\boxed{e^{\int p(x)\,dx} = g(x)}$.

multiply:

$$\underbrace{e^{\int p(x)\,dx}\, y' + p(x)\, e^{\int p(x)\,dx}\, y} = e^{\int p(x)\,dx}\, q(x).$$

$$\left(e^{\int p(x)\,dx}\, y\right)' = g(x)\, q(x)$$

$$(g(x)\, y)' = g(x)\, q(x)$$

$$g(x)\, y = \int g(x)\, q(x)\, dx + c$$

$$y = \frac{1}{g(x)} \int g(x)\, q(x)\, dx + \frac{c}{g(x)}$$