Math 330 ODEs Fall 15 Sample midterm 2

(1) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

compute and show your work:

(a) Find eigenvalues of A.
(b) Find corresponding eigenvectors.

(2) Let $\mathbf{e}_1 = (0, -1, 3)$, $\mathbf{e}_2 = (-2, -2, 1)$ and $\mathbf{e}_3 = (3, -6, 12)$.

(a) Verify that they form a basis for $\mathbb{R}^3$;
(b) Let $\mathbf{u} = (13, -12, 28)$, expand it as a linear combination of above basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$;
(c) Find the angle between $\mathbf{e}_1$ and $\mathbf{e}_3$, and compute the volume of the parallelepiped spanned by $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

(3) Solve the IVP: $\mathbf{X}' = A\mathbf{X}$, where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and

$$\mathbf{X}(\mathbf{0}) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

(4) Solve the following IVP: $\mathbf{X}' = A\mathbf{X}$, where

$$A = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix},$$

and

$$\mathbf{X}(\mathbf{0}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(5) Solve the following linear system, show your work.

$$\begin{cases} x - 3y + z = 4 \\ 2x - 8y + 8z = -2 \\ 6x - 3y + 15z = -9 \end{cases}$$

(6) Determine the solution space for the homogeneous system:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 = 0 \\ 2x_1 + 4x_2 + 6x_3 + 2x_4 + 6x_5 = 0 \\ 3x_1 + 6x_2 + 18x_3 + 9x_4 + 9x_5 = 0 \\ 4x_1 + 8x_2 + 12x_3 + 10x_4 + 12x_5 = 0 \\ 5x_1 + 10x_2 + 24x_3 + 11x_4 + 15x_5 = 0 \end{cases}$$

(7) Find the equilibrium points and classify their stability properties for the folowing differential equations.
 (a) $y' = y(y-1)(y-2)$
 (b) $y' = -2\tan^{-1}(y/(1+y^2))$
 (c) $y' = y(1-y^2)$

(8) Sketch the phase portraits for
 (a) $x'' - x = 0$
 (b) $x'' - x + x^3 = 0$

(9) Find the linearizations at the origin of the following systems.
 (a) $x' = x + x^2 + xy^2$, $y' = y + y^{3/2}$
 (b) $x' = x^2 e^y$, $y' = y(e^x - 1)$

(10) Show that the system $x' = e^{x+y} - y$, $y' = -x + xy$ has only one fixed point. Find the linearization of the system at this point and discuss its stability.

(11) In a simple model of the national economy, national income $I(t)$ and consumer spending $C(t)$ satisfy

$$I' = I - \alpha C$$

$$C' = \beta(I - C - G),$$

where $\alpha > 1$, $\beta \geq 1$ are constants and $G(t)$ is the rate of gevernment spending.
 (a) Show that if $G(t) = G_0$ is constant, then there is an equilibrium state. Classify the equilibrium state and show that the economy oscillates when $\beta = 1$.

(b) Consider the situation when government spending is related to national income by the rule $G = G_0 + kI$, where $k > 0$. Show there is no equilibrium state if $k > (\alpha - 1)/\alpha$. How does the economy behave?

(c) Discuss what happens if $G = G_0 + kI^2$.