Math 330 ODEs Fall 15 Sample midterm 1

(1) Show that the general solution of

$$y' - y = e^{\mu x}, \mu \neq 1, \qquad (*)$$

can be written as

$$y(x) = Ae^x + \frac{e^x - e^{\mu x}}{\mu - 1}$$

Take the limit as $\mu \to 1$ to find the general solution to $(*)$ when $\mu = 1$.

(2) Solve
(a) $y' - y = 2e^x$
(b) $y' + 2y = xe^{-2x}$
(c) $y' = (e^y - x)^{-1}$
(d) $y' \tan x + y = 1$
(e) $y'x \sin x + (\sin x + x \cos x)y = xe^x$
[Hint: for (c) and (d) guess a solution $f(x)$, then look for a general solution $f(x) + u(x)$.]

(3) Sketch the flow vectors for the equation

$$\frac{dy}{dx} = xy.$$

Find and sketch the family of solutions determined by this equation.

(4) Sketch the flow vectors for the equation

$$\frac{dy}{dx} = \frac{x - y}{x + y}.$$

Using the substitution $y = ux$, find and sketch the family of curves dtermined by the equation.

(5) Water flows into a cylindrical bucket of depth H and cross-sectional area A, at a volume flow rate Q. There is a hole in the bottom of the bucket of cross sectional area $a < A$. When the water level above the hoel is h, the flow rate out of the hole is $a\sqrt{2gh}$, where g is the gravitaional acceleration. Derive an equation for $\frac{dh}{dt}$. Find the equilibrium depth h_e of water. Suppose the water is turned off when the bucket is full, how long does it take to empty?

(6) Find the general solution of
(a) $y'' + 5y' + 6y = 3e^{-2x} + e^{3x}$
(b) $y'' - 2y' + y = (x - 1)e^x$

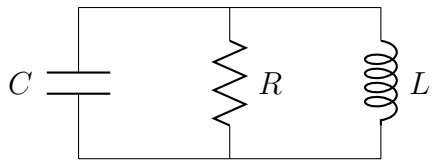
(7) Given the solution $u(x)$, find a second solution to the following equations
   (a) $x(x+1)y'' + (x-1)y' - y = 0, u(x) = (x+1)^{-1}$
   (b) $xy'' - y' - 4x^3y = 0, u(x) = e^{x^2}$

(8) Find the solution of
$$y'' - y' - 2y = 0$$
which satisfies $y(0) = 1$, and which is bounded as $x \to \infty$.

(9) A large oil tanker of mass W floats on a sea of density ρ. Suppose the tanker is given a small downward displacement z. The upward force is equal to the weight of water displaced (Archimedes principle). If the cross-sectional area A of the tanker at the water's surface is constant show that this upward force is $g\rho Az$, and hence that
$$z'' + \frac{g\rho A}{W} z = 0.$$
Suppose now that a small mouse jumps on the deck of the tanker with frequency $\omega = (g\rho A/W)^{1/2}$, and exerts a force $m \sin \omega t$. Show that the tanker will eventually sink.

In practice, as the vertical motion of the tanker increases, waves will be generated. Suppose they produce an additional damping kz'. Discuss the motion for a range of values of k.

(10) In the circuit shown $L = 2R^2C$ and at time $t = 0$ there is a charge Q_0 on C and zero current flowing through L.



Show that at a later time the charge on C is $Q_0 e^{-kt}(\cos kt - \sin kt)$, where $k = 1/2RC$.

[The potential drop V across an inductance L is $L\frac{dI}{dt}$, and across a capacitance C is Q/C, where I is the current.]