Math 330 Differential Equations Fall 15 Final a

Name: Solutions

- I will count your best 8 of the following 10 questions.
- You may use yor textbooks and notes, but no electronic devices.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) (10 points) Find the general solution to the following differential equation.

$$y' = \frac{x + 2y}{x}$$

sub $y = ux$

$y' = u'x + u$

$$u'x + u = \frac{x + 2ux}{x} = 1 + 2u$$

$$u'x = 1 + u$$

$$\int \frac{du}{1+u} = \int \frac{dx}{x}$$

$$\ln|1+u| = \ln|x| + c$$

$$1 + u = Ax$$

$$\frac{y}{x} = u = Ax - 1$$

$$y = Ax^2 - x$$

(2) (10 points) Find the solution to

$$y'' - 9y = e^{-3x}$$

with $y(0) = 1$ which stays bounded as $x \to \infty$.

homogeneous : $y'' - 9y = 0$ try $y = e^{\lambda x}$ $e^{\lambda x}(\lambda^2 - 9) = 0$

$(\lambda-3)(\lambda+3)$

$\lambda = \pm 3.$

$y_h = c_1 e^{3x} + c_2 e^{-3x}$

particular solution: try $y = Axe^{-3x}$

$y' = Ae^{-3x} - 3Axe^{-3x}$

$y'' = (-3e^{-3x} - 3e^{-3x} + 9xe^{-3x})A.$

plug in: $A(-6e^{-3x} + \cancel{9xe^{-3x}} - \cancel{9xe^{-3x}}) = e^{-3x}$ $\Rightarrow$ $-6A = 1$ $A = -1/6$

general solution:

$c_1 e^{3x} + c_2 e^{-3x} - \frac{1}{6}xe^{-3x}$

bounded $\Rightarrow$ $c_1 = 0$

$y(0) = 1 = c_2$

solution: $e^{-3x} - \frac{1}{6}xe^{-3x}$

(3) (10 points) Find the eigenvalues and eigenvectors for the following matrix.

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = (\lambda-3)\lambda+2 = \lambda^2-3\lambda+2$$

$$= (\lambda-2)(\lambda-1) \quad \lambda = 1, 2$$

$\lambda = 1$: $\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2$: $\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(4) (10 points) Consider the following differential equation.

$$X' = AX, \text{ where } A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

(a) Find the general solution in the form $\Omega(t)C$, where $\Omega(t)$ is the fundamental matrix solution and $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

(b) Find Ω^{-1}.

You may use your solution to the previous question.

a)

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} = \underbrace{\begin{bmatrix} e^{t} & e^{2t} \\ e^{t} & 2e^{2t} \end{bmatrix}}_{\Omega} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

b)

$$\Omega^{-1} = \frac{1}{2e^{3t} - e^{3t}} \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -e^{t} & e^{t} \end{bmatrix} = \cancel{\tfrac{1}{2}} \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

(5) (10 points) Find the general solution to the following differential equation by looking for a solution of the form $X(t) = \Omega(t)U(t)$.

$$X' = AX + F(t), \text{ where } A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, F(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

You may use your solution to the previous question.

plug in:

$$\cancel{\Omega' U} + \Omega U' = \cancel{A\Omega U} + F \qquad U' = \Omega^{-1}F$$

$$U = \int \Omega^{-1} F \, dt$$

$$\Omega^{-1}F = \begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4e^{-t} - e^{-t} \\ -2e^{-2t} + e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-t} \\ -e^{-2t} \end{bmatrix}$$

$$\int \Omega^{-1} F \, dt = \begin{bmatrix} -3e^{-t} \\ \frac{1}{2}e^{-2t} \end{bmatrix}$$

general solution: $\Omega C + \Omega \int \Omega^{-1} F \, dt$

$$= \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}\begin{bmatrix} -3e^{-t} \\ \frac{1}{2}e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -5/2 \\ -2 \end{bmatrix}$$

(6) (10 points) Find the equilibrium solutions and investigate their stability for the following differential equation.

$$x'' = xe^x - ex + (x')^2$$

$x' = y$

$y' = xe^x - e\cancel{x} + y^2$

$X' = F(X)$ solve $F(X) = 0$: $y = 0$

$x(e^x - e) = 0$

$x = 0,\ x = 1$.

$(0,0)$ and $(1,0)$

$$DF = \begin{bmatrix} 0 & 1 \\ e^x + xe^x - e & 2y \end{bmatrix}$$

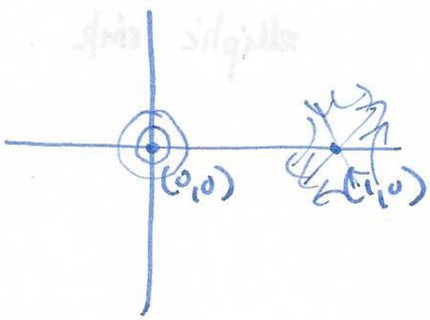
$$DF\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1-e & 0 \end{bmatrix}$$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ 1-e & -\lambda \end{vmatrix} = \lambda^2 - (1-e) = 0$

$\lambda = \pm\sqrt{e-1}\, i$

~~saddle~~ elliptic

$$DF\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \underbrace{e+e-e}_{=e} & 0 \end{bmatrix}$$ eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ e & -\lambda \end{vmatrix} = \lambda^2 - e = 0$

$\lambda = \pm\sqrt{e}$ saddle



(7) (10 points) Find the equilibrium solutions and investigate their stability for the following system of differential equations.

$$x' = xy + x$$
$$y' = x - y$$

$x' = F(x)$

solve $F(x) = 0$ $\quad x(y+1) = 0 \quad \Rightarrow x = 0,\ y = -1 \qquad (0,0)$ and $(-1,-1)$

$y = x$

$$DF = \begin{bmatrix} y+1 & x \\ 1 & -1 \end{bmatrix}$$

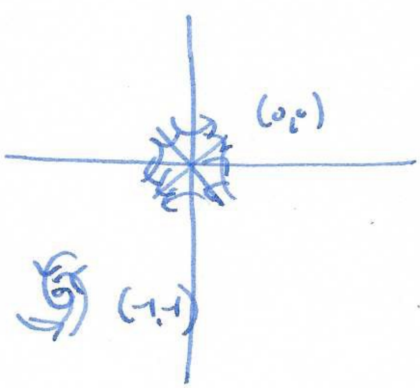
$$DF\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$ eigenvalues $1, -1$ saddle

$$DF\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$ eigenvalues $\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda(1+\lambda) + 1$

$\lambda^2 + \lambda + 1$

$\lambda = \dfrac{-1 \pm \sqrt{1-4}}{2}$

elliptic sink.



(8) (10 points) Find the inverse Laplace transform for the following function.

$$F(s) = \frac{4e^{-3s}}{s^2 + 2s + 2} = 4e^{-3s} \frac{1}{(s+1)^2+1}$$

$$\sin(at) \xmapsto{L} \frac{a}{s^2+a^2} \Rightarrow \sin(t) \xmapsto{L} \frac{1}{s^2+1}$$

$$e^{at} f(t) \xmapsto{L} F(s-a) \Rightarrow e^{-t}\sin(t) \longmapsto \frac{1}{(s+1)^2+1}$$

$$H(t-a) f(t-a) \xmapsto{L} e^{-as} F(s)$$

$$\Rightarrow L^{-1}\left(4e^{-3s} \frac{1}{(s+1)^2+1}\right) = 4H(t-3)\, e^{-(t-3)} \sin(t-3)$$

(9) (10 points) Find the Laplace transform of the function.

$$f(t) = \begin{cases} 0 & t < 3 \\ e^{-2t} & t \geq 3 \end{cases}$$

$f(t) = H(t-3)e^{-2t} = H(t-3)e^{-2(t-3+3)} = e^{-6}H(t-3)\underbrace{e^{-2(t-3)}}$

$H(t-a)f(t-a) \overset{L}{\longmapsto} e^{-as}F(s)$, $e^{at} \overset{L}{\longmapsto} \frac{1}{s-a}$ $\quad f(t) = e^{-2t}$

$L(f(t)) = e^{-6} \cdot e^{-3s} \frac{1}{s+2}$

(10) (10 points) Use the Laplace transform to solve the following IVP.

$$y'' + 5y' + 6y = \begin{cases} 0 & t < 3 \\ e^{-2t} & t \geq 3 \end{cases} \quad y(0) = 0,\ y(0) = 0$$

Hint: you may use your answer to the previous question.

$$s^2Y - s\underset{0}{y(0)} - \underset{0}{y'(0)} + 5(sY - \underset{0}{y(0)}) + 6Y = \frac{e^{-6-3s}}{s+2}$$

$$Y(s^2+5s+6) = Y(s+3)(s+2) = \frac{e^{-6-3s}}{s+2}$$

$$Y = \frac{e^{-6-3s}}{(s+2)^2(s+3)} \qquad \frac{1}{(s+2)^2(s+3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3}.$$

$$1 = A(s+2)(s+3) + B(s+3) + C(s+2)^2$$

$$s=-3: \quad 1 = C$$

$$s=-2: \quad 1 = B$$

$$s=0: \quad 1 = 6A + \underbrace{3B}_{3.} + \underbrace{4C}_{4} \qquad A = -1$$

$$Y = e^{-6-3s}\left[\underbrace{\frac{-1}{s+2}}_{\downarrow\, -e^{-2t}} + \underbrace{\frac{1}{(s+2)^2}}_{\downarrow\, te^{-2t}} + \underbrace{\frac{1}{s+3}}_{\downarrow\, e^{-3t}}\right]$$

$$y(t) = e^{-6}H(t-3)\left[-e^{-2(t-3)} + (t-3)e^{-2(t-3)} + e^{-3(t-3)}\right].$$