Math 330 Differential Equations Fall 2015 Sample Final

1. Solve the following differential equations:

   (a) $y' \tan(y) - \dfrac{x^2}{1+x^2} = 1$

   (b) $y'' - 2y' + 2y = e^t \sin(t)$

   (c) $y' = \dfrac{3t}{y + t^2 y}, \qquad y(0) = 1$

   (d) $(x-1)y' + y = x^2 - 2, \qquad y(2) = 1$

2. Compute the Laplace transforms or the inverse Laplace transforms (if the variable is s) for the following functions:

   (a) $f(t) = (t^2 - 2t + 1)(e^{-t} - 1)$

   (b) $f(t) = te^{-t}\cos(3t)$

   (c) $f(t) = \begin{cases} -t & t < 4; \\ t^2 + 1 & t \geq 4 \end{cases}$

   (d) $F(s) = \dfrac{3e^{-2s}}{s^2 - 9}$, your answer can't use sinh or cosh functions;

   (e) $F(s) = \dfrac{-2s+1}{s^2 + 4s + 13}$

3. Solve the following IVP.

$$y'' + 3y' + 2y = f(t), \qquad y(0) = 1, \quad y'(0) = 1.$$

   where

$$f(t) = \begin{cases} e^t & t < 2; \\ 0 & t \geq 2 \end{cases}$$

4. Use Laplace transform to solve the following initial value problem:

$$y'' + 4y' + 4y = 2\delta(t+1), y(0) = 0, y'(0) = -1.$$

5. Solve the IVP: $\mathbf{X}' = A\mathbf{X}$, where

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \text{ and } \mathbf{X}(\mathbf{0}) = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

6. Solve the following nonhomogeneous system: $\mathbf{X}' = A\mathbf{X} + \mathbf{f}(t)$, where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \text{ and } \mathbf{f}(\mathbf{t}) = \begin{pmatrix} 3t \\ 2 \end{pmatrix}.$$

7. Find an expression for a matrix (with respect to the standard basis) for the linear map from $\mathbb{R}^3 \to \mathbb{R}^3$ which expands by a factor of 3 in the direction $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and rotates by $\pi/4$ (counterclockwise using the right hand rule) about this direction.

8. Find bases for $U + V$ and $U \cap V$, where

$$V = \text{span}\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\}, \quad W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\}.$$

You may use the fact that

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

9. Find the equilibrium solutions for $x''' = x - x^2 - x'x''$ and investigate their stability.

10. Find the equilibrium solutions and investigate their stability for

$$x' = xy - 1$$

$$y' = y - x$$