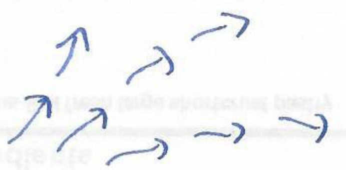
$$\operatorname{curl}(\underline{F}) = \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{\text{Green's thm!}} \right\rangle$$

__Example__ $\underline{F} = \underline{x} = \langle x, y, z \rangle$, $\underline{\nabla} \times \underline{F} = \underline{0}$

$$\underline{F} = \left\langle xy, e^x, \frac{y}{z} \right\rangle \qquad \underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & e^x & \frac{y}{z} \end{vmatrix} = \left\langle \frac{1}{z} - 0, -0 + 0, e^x - x \right\rangle$$

__intuition__ what does curl($\underline{F}$) measure?

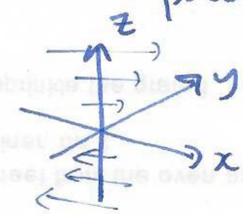
fluid flow:

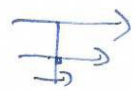


imagine flowing along the vector field.

$\operatorname{curl}(\underline{F}) = \left(\frac{1}{2}\right)$ angular velocity of small paddle wheel

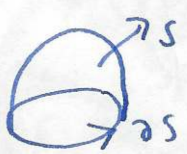
__Example__ $\underline{F} = \langle y, 0, 0 \rangle$ $\quad \underline{\nabla} \times \underline{F} = \langle 0, 0, 1 \rangle$





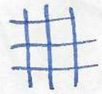
rotational force

__Intuition__

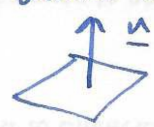


$$\oint_{\partial S} \underline{F} \cdot d\underline{s} = \iint_S \operatorname{curl}(\underline{F}) \cdot d\underline{S}$$

chop into small bits



same calculation as last time

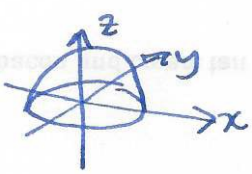


$\underline{n} \approx (0, 0, 1)$

$$\underline{\nabla} \times \underline{F} \cdot \underline{n} \approx \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_3}{\partial y}\right) \square$$

__Example__ $\underline{F} = \langle y, x, x + z \rangle$

S = upper hemisphere



$x^2 + y^2 + z^2 = 1$, $z \geq 0$

$\int_{\partial S} \underline{F} \cdot d\underline{s}$ $\quad \underline{c}(\theta) = (\cos\theta, \sin\theta, 0)$

$\quad \underline{c}'(\theta) = (-\sin\theta, \cos\theta, 0)$

$$\oint_{\partial S} \underline{F} \cdot d\underline{s} = \int_0^{2\pi} (-\sin\theta, \cos\theta, \cos\theta) \cdot (-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi.$$

$\iint_S \mathrm{curl}(\underline{F})\cdot d\underline{S}$ $\quad \underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & x+z \end{vmatrix} = \langle 0, -1, 2 \rangle$

parameterization: $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ $\quad 0 \le \theta \le 2\pi$, $\ 0 \le \phi \le \pi$

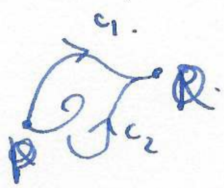
$$\int_0^{\pi/2}\int_0^{2\pi} (0,-1,2)\cdot\underline{n}\, d\theta\, d\phi \qquad \frac{\partial\Phi}{\partial\theta} = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

$$\underline{n} = \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin^2\theta\sin\phi\cos\phi + \cos^2\theta\cos\phi\sin\theta \rangle$$

$$= \sin\phi\,\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle$$

$$\int_0^{\pi/2}\int_0^{2\pi} \underbrace{-\sin\theta\sin^2\phi}_{0.} + \underbrace{2\sin\phi\cos\phi}_{\sin 2\phi}\, d\theta\, d\phi = 2\pi.$$

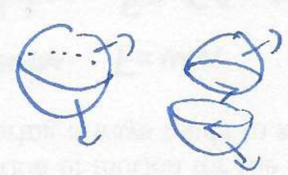


<u>recall</u> $F = \nabla\phi$ then $\int_{c_1} \underline{F}\cdot d\underline{s} = \int_{c_2} \underline{F}\cdot d\underline{s} = \phi(Q) - \phi(P)$

c_1, c_2 any paths from P to Q. (path independence for gradient vector fields)

<u>Stoke's thm</u>: $\oint_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S \mathrm{curl}(\underline{F})\cdot d\underline{S}$ $\leftarrow$ surface independence for curl vector fields

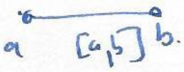
in particular, if S is closed ($\partial S = \phi$) then $\iint_S \mathrm{curl}(\underline{F})\, d\underline{S} = 0$



$S_1 \cup S_2$ with $\partial S_1 = -\partial S_2$ (reverse orientation)

§17.3 Divergence Theorem

integral of a derivative on a domain = integral of function over boundary

<u>recall</u>: FTC: $\int_a^b f'(x)\,dx = f(b) - f(a)$  $\partial[a,b] = b - a$.

line integrals: $\int_C \underline{\nabla}\phi\cdot ds = \phi(Q) - \phi(P)$ 

Stokes's Thm $\iint_S \text{curl}(\underline{F})\cdot d\underline{S} = \int_{\partial S} \underline{F}\cdot d\underline{s}$

Divergence Thm: $\iiint_W \text{div}(\underline{F})\cdot dV = \iint_{\partial W} \underline{F}\, d\underline{S}$

outward pointing normal

Defn divergence of $\underline{F}$ is $\text{div}(\underline{F}) = \underline{\nabla}\cdot\underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ (scalar!)

Example $\underline{F} = \langle x^2+y^2, xyz, e^{xyz} \rangle$ $\underline{\nabla}\cdot\underline{F} = 2x + xz + xye^{xyz}$

Intuition:

small ball B, so $\text{div}(F)$ approx constant

then flux across $S = \iiint_B \text{div}(F)\,dV \simeq \text{div}(F)\,\text{vol}(B)$

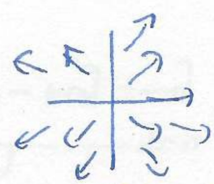
locally: flux $\simeq$ divergence . vol

- $\text{div}(\underline{F}) > 0$ at a point: fluid flows out "source"

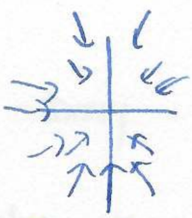
- $\text{div}(F) < 0$: fluid flows in "sink"

- $\text{div}(\underline{F}) = 0$: "incompressible flow"

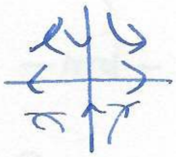
Examples (2d)



$\underline{F} = \langle x, y \rangle$ $\underline{\nabla}\cdot\underline{F} = 2$



$\underline{F} = \langle -x, -y \rangle$
$\nabla\cdot F = -2$



$\underline{F} = \langle x, -y \rangle$ $\underline{\nabla}\cdot\underline{F} = 0$

Example (verify divergence thm)

$\underline{F} = \langle y, yz, z^2 \rangle$ $S: x^2+y^2 = 4$, $0 \le z \le 5$ $\underline{\nabla}\cdot\underline{F} = 0 + z + 2z = 3z$

$\iiint_W 3z\, dV = 150\pi$ (use cylindrical coordinates).