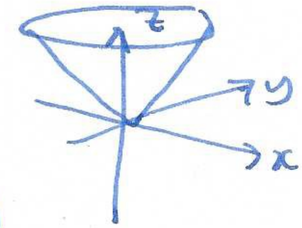
surface area: $\text{area}(S) = \iint_R \|\underline{n}(u,v)\|\, du\, dv$

How to integrate a scalar function over S:

$$\iint_S f(x,y,z)\, dS = \iint_R f(x,y,z)\, \|\underline{n}(u,v)\|\, du\, dv$$

↑ S with same choice of parameterization $\phi(u,v)$

<u>Example</u> find surface area of a cone



$\phi(\theta, t) = (t\cos\theta, t\sin\theta, t)$

$0 \leq \theta \leq 2\pi$
$0 \leq t \leq 1$

find $\underline{n}$: $\frac{\partial \phi}{\partial \theta} = (-t\sin\theta, t\cos\theta, 0)$

$\frac{\partial \phi}{\partial t} = (\cos\theta, \sin\theta, 1)$

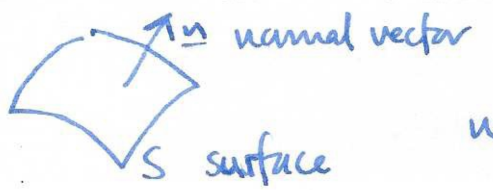
$$\underline{n} = \frac{\partial \phi}{\partial \theta} \times \frac{\partial \phi}{\partial t} = \begin{vmatrix} i & j & k \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \langle t\cos\theta, t\sin\theta, -t \rangle$$

$$\|\underline{n}\| = \sqrt{t^2\cos^2\theta + t^2\sin\theta + t^2} = \sqrt{2}\, t$$

$$\text{area} = \int_0^{2\pi} \int_0^1 1 . \sqrt{2}\, t\, dt\, d\theta = \left[\sqrt{2} \frac{t^2}{2} \right]_0^1 = \frac{\sqrt{2}}{2}$$

$$\int_0^{2\pi} \frac{\sqrt{2}}{2}\, d\theta = \sqrt{2}\pi .$$

<u>§16.5 Vector integrals over surfaces</u>

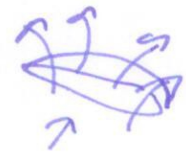


integrate a vector field $\underline{F}$ over S

notation: $\iint_S \underline{F} . \underline{dS} = \iint_S (\underline{F} . \hat{\underline{n}})\, dS$ $\quad \hat{\underline{n}}$ = unit normal this is called the flux of $\underline{F}$ across S

in terms of a parameterization $\phi(u,v)$ for S: $\iint_D \underline{F}(\phi(u,v)) \cdot \underline{n}(u,v)\, du\, dv$

Example $\underline{F}$ = fluid flow



$\iint_S \underline{F} \cdot \underline{dS}$ = rate of fluid flowing across the surface

Electricity $\underline{E}$ electric field
$\underline{B}$ magnetic field

Faraday's law of induction: $\int_C \underline{E} \cdot \underline{ds} = -\frac{d}{dt} \iint_S \underline{B} \cdot \underline{dS}$

Warning: can only integrate vector functions over orientable surfaces

orientable  non-orientable:  Möbius band

Example find $\iint_S \underline{F} \cdot \underline{dS}$

$\underline{F} = \langle 0, 0, x \rangle$

$S: \Phi(u,v) = (u^2, v, u^3 - v^2)$ $\quad 0 \le u \le 1$, $0 \le v \le 1$

$\frac{\partial \phi}{\partial u} = \langle 2u, 0, 3u^2 \rangle$

$\frac{\partial \phi}{\partial v} = \langle 0, 1, -2v \rangle$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix} = \langle -3u^2, 4uv, 2u \rangle$$

$$\iint_S \underline{F} \cdot \underline{dS} = \int_0^1 \int_0^1 \underline{F}(\phi(u,v)) \cdot \underline{n}(u,v)\, du\, dv$$

$$= \int_0^1 \int_0^1 \langle 0, 0, u^2 \rangle \cdot \langle -3u^2, 4uv, 2u \rangle\, du\, dv = \int_0^1 \int_0^1 2u^3\, du\, dv$$

$\left[\frac{1}{2} u^4 \right]_0^1 = \frac{1}{2}$ $\qquad \int_0^1 \frac{1}{2}\, dv = \frac{1}{2}.$

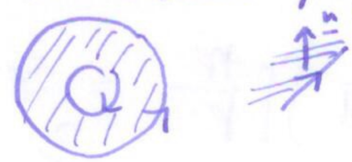
§17.1 Green's Theorem

recall $\oint_C \underline{\nabla f} \cdot \underline{ds} = 0$

spose $\underline{F} \neq \nabla f$?

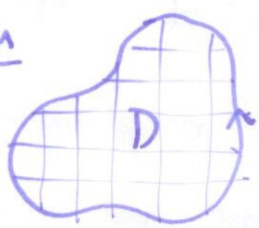
orientation convention:

counter clockwise / right hand rule

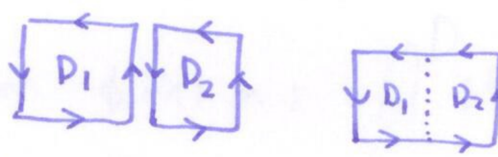


Thm [Green's Thm] $D \subseteq \mathbb{R}^2$ domain such that $\partial D = C$ simple closed curve oriented clockwise. Then $\oint_C \underline{F} \cdot \underline{ds} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA$ $\quad \underline{F} = \langle F_1, F_2 \rangle$ □

intuition



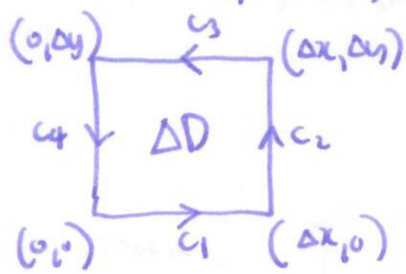
note:



$$\oint_{\partial D_1} + \oint_{\partial D_2} = \oint_{\partial (D_1 \cup D_2)}$$

so suffices to check on a small square where $\underline{F}$ is almost linear

$\underline{F} = \langle px + qy, rx + sy \rangle$



c_1: $\int_0^{\Delta x} pt + q.0 \, dt = \left[\frac{1}{2} pt^2 \right]_0^{\Delta x} = \frac{1}{2} p \Delta x^2$

c_2: $\int_0^{\Delta y} r\Delta x + st \, dt = \left[r\Delta x t + \frac{1}{2} st^2 \right] = r \Delta x \Delta y + \frac{1}{2} s \Delta y^2$

c_3: $-\int_0^{\Delta x} pt + q\Delta y \, dt = -\left[\frac{1}{2} pt^2 + q\Delta y t \right]_0^{\Delta x} = -\frac{1}{2} p \Delta x^2 - q \Delta x \Delta y$

c_4: $-\int_0^{\Delta y} r.0 + st \, dt = -\left[\frac{1}{2} st^2 \right]_0^{\Delta y} = -\frac{1}{2} s \Delta y^2$

$$\oint_{\Delta D} \underline{F}\cdot\underline{ds} = r\Delta x\Delta y - q\Delta x\Delta y = (r-q)\Delta A = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\Delta A.$$

$$\oint_{D} \underline{F}\cdot\underline{ds} = \sum (r-q)\Delta A = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \quad \square.$$

<u>Example</u> $\oint_C \langle xy^2, x\rangle \cdot \underline{ds}$ C = unit circle w/ clockwise orientation

<u>Line integral</u> $c(\theta) = (\cos\theta, \sin\theta) \quad 0 \leq \theta \leq 2\pi$

$c'(\theta) = (-\sin\theta, \cos\theta)$

$$\int_0^{2\pi} (\cos\theta\sin^2\theta, \cos\theta)\cdot(-\sin\theta, \cos\theta)\, d\theta = \int_0^{2\pi} -\cos\theta\sin^3\theta + \cos^2\theta \, d\theta$$

$$= \left[-\frac{\sin^4\theta}{4} + \frac{1}{2}\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi$$

<u>double integral</u> $\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_D 1 - 2xy \, dA = \iint_D 1\, dA = \text{area} = \pi.$

$\iint_D xy\, dA = 0$ by symmetry

<u>Multiple boundary components</u> : orient components.

think:

<u>§17.2 Stoke's Theorem</u> (3d version of Green's Thm)

<u>Thm</u> $\oint_{\partial S} \underline{F}\cdot\underline{ds} = \iint_S \underline{\nabla}\times\underline{F}\cdot\underline{dS}$

S oriented surface

∂S boundary (induced orientation)

<u>remark</u> if S is <u>closed</u> ($\partial S = \emptyset$) then

$\oint_{\partial S = \emptyset} \underline{F}\cdot\underline{ds} = 0$