③ find the potential function

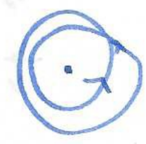
$f(x,y) = \tan^{-1}(y/x)$ $(= \theta$ in polars!$)$

check: $\dfrac{\partial f}{\partial x} = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{-y}{x^2} = \dfrac{-y}{x^2+y^2}$.

$\dfrac{\partial f}{\partial y} = \dfrac{1}{1+(y/x)^2} \cdot \dfrac{1}{x} = \dfrac{x}{x^2+y^2}$ so $\nabla f = \underline{F}$.

④ $\tan^{-1}(\frac{y}{x})$ not defined at $(0,0)$ and $D = \mathbb{R}^2 \setminus (0,0)$ not simply connected

Fact $\oint_{\underline{c}} \underline{F} \cdot \underline{ds} = 2\pi n$ n = winding number.



$n=2$.

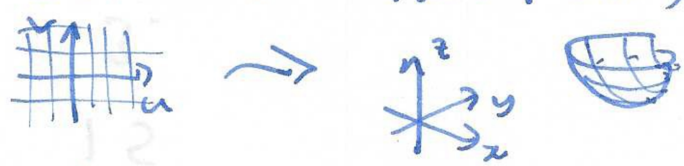
§16.4 Parameterized surfaces and surface integrals

recall: parameterized curve $\underline{c}(t) : \mathbb{R} \to \mathbb{R}^3$

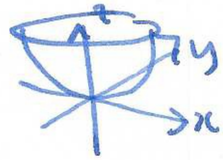
$t \mapsto (x(t), y(t), z(t))$

parameterized surface $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$

$(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$



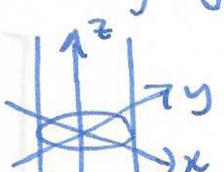
Examples ① paraboloid $z = x^2+y^2$



$\Phi(u,v) = (u, v, u^2+v^2)$

note: we can parameterize any graph $z = f(x,y)$ by $(u,v) \mapsto (u, v, f(u,v))$

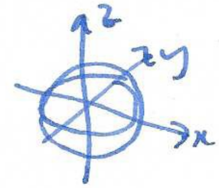
② cylinder $x^2+y^2 = 1$



$(\theta, z) \longmapsto (\cos\theta, \sin\theta, z)$

$0 \leq \theta \leq 2\pi$

③ sphere $x^2+y^2+z^2=1$

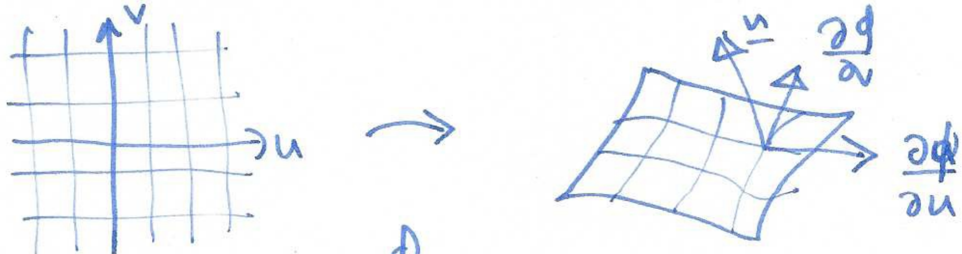


spherical coords!
$\rho = 1$

$(\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

coordinate lines



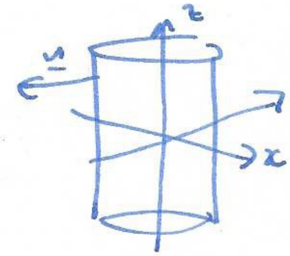
$$(u,v) \xmapsto{\phi} (x(u,v), y(u,v), z(u,v))$$

$\frac{\partial \phi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$ = tangent vector to surface in u-direction

$\frac{\partial \phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$ = tangent vector to surface in v-direction

Q: how do we find the normal vector to the surface?

A: $\underline{n} = \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}$



Example cylinder $(\theta, z) \mapsto (\cos\theta, \sin\theta, z)$

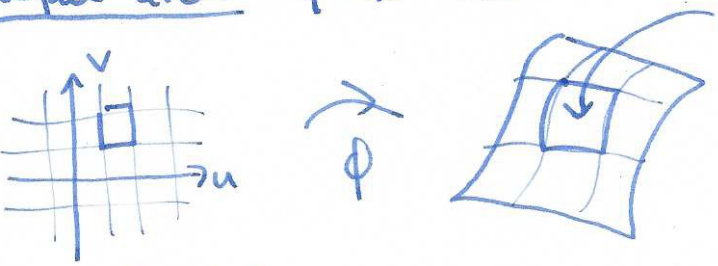
$\frac{\partial \phi}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$

$\frac{\partial \phi}{\partial z} = (0, 0, 1)$

$$\underline{n} = \frac{\partial \phi}{\partial \theta} \times \frac{\partial \phi}{\partial z} = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$

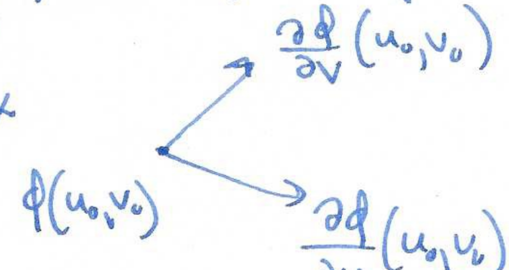
Surface area $\phi : \mathbb{R}^2 \to \mathbb{R}^3$

need scaling function for this piece



$\Delta A = \Delta u \Delta v$

linear approx



we can use area of parallelogram $= \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| = \| \underline{n}(u_0, v_0) \|$