so want $\lim_{n\to\infty} \sum f(t_k) \int_{t_k}^{t_{k+1}} \|\underline{c}'(t)\|\, dt \rightsquigarrow \int_a^b f(t)\|\underline{c}'(t)\|\, dt$.

<u>Notation</u> $ds = \|\underline{c}'(t)\|\, dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\; dt$

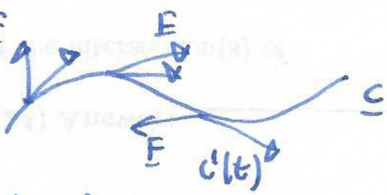
<u>Example</u> find $\int_C (x+y+z)\, ds$ where C is the helix $\underline{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$

$\underline{c}'(t) = (-\sin t, \cos t, 1)$

$= \int_0^{2\pi} (\cos t + \sin t + t)(\sin^2 t + \cos^2 t + 1)^{1/2}\, dt$

$= \sqrt{2} \int_0^{2\pi} \cos t + \sin t + t\, dt = \sqrt{2}\left[-\sin t + \cos t + \frac{1}{2}t^2\right]_0^{2\pi} = 2\sqrt{2}\pi^2$

<u>Vector line integrals</u> $\int_C \underline{F}.d\underline{s}$



intuition: integral of tangential component of $\underline{F}$ along c,

i.e. $\simeq \sum_{i=1}^{n} \underline{F}(x,y,z) . \underline{c}'(t)\,\Delta t_i$

<u>Defn</u> let T = unit tangent vector to $\underline{c}$: $\underline{T}(t) = \frac{\underline{c}'(t)}{\|\underline{c}'(t)\|}$

then $\int_C \underline{F}.d\underline{s} = \int_C (\underline{F}.\underline{T})\, ds$

If $\underline{c}$ has parameterization $c(t)$:

$$\int_C \underline{F}.d\underline{s} = \int_a^b \underline{F}(\underline{c}(t)) . \frac{\underline{c}'(t)}{\|\underline{c}(t)\|} \|c'(t)\|\, dt$$

$$\boxed{\int_C \underline{F}.d\underline{s} = \int_a^b \underline{F}(\underline{c}(t)) . \underline{c}'(t)\, dt}$$

Alternate notation $\int_C \underline{F}\cdot\underline{ds}$ $\quad \underline{F} = \langle F_1, F_2, F_3 \rangle$

$$= \int_C F_1\,dx + F_2\,dy + F_3\,dz$$

to evaluate this with parameterization $\underline{c}(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_C \underline{F}\cdot\underline{ds} = \int_a^b \left(F_1(\underline{c}(t))\frac{dx}{dt}\cancel{dt} + F_2(\underline{c}(t))\frac{dy}{dt} + F_3(\underline{c}(t))\frac{dz}{dt} \right) dt$$ $\leftarrow$ same as before.

Example (2d) integrate $\underline{F} = \langle 2y, -3 \rangle$ over the ellipse

$$\underline{c}(t) = (4 + 3\cos\theta,\ 3 + 2\sin\theta) \qquad 0 \leq \theta \leq 2\pi$$

$$\underline{c}'(t) = (-3\sin\theta,\ 2\cos\theta)$$

$$\int_0^{2\pi} \underline{F}\cdot\underline{ds} = \int_0^{2\pi} \langle 6 + 4\sin\theta, -3 \rangle \cdot \langle -3\sin\theta, 2\cos\theta \rangle\, d\theta$$

$$= \int_0^{2\pi} -18\sin\theta - 12\sin^2\theta - 6\cos\theta \ d\theta = \int_0^{2\pi} -12\sin^2\theta \ d\theta$$

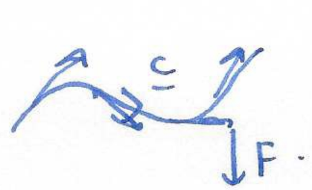
$$= \int_0^{2\pi} -6 + 6\cos 2\theta \ d\theta = -6\int_0^{2\pi} d\theta = -12\pi.$$

useful properties $\int_C (\underline{F} + \underline{G})\cdot\underline{ds} = \int_C \underline{F}\cdot\underline{ds} + \int_C \underline{G}\cdot\underline{ds}$

$$\int_C k\underline{F}\cdot\underline{ds} = k\int_C \underline{F}\cdot\underline{ds}$$

reverse orientation: $\int_C \underline{F}\cdot\underline{ds} = -\int_{-C} \underline{F}\cdot\underline{ds}$

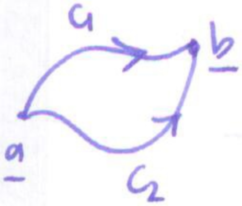
Physical interpretation



work done $W = \int_C \underline{F}\cdot\underline{ds}$.

§16.3 Conservative vector fields

In general $\int_C \underline{F}.\underline{ds}$ depends on the path C, not just the endpoints.



$$\int_{C_1} \underline{F}.\underline{ds} \neq \int_{C_2} \underline{F}.\underline{ds}$$

However for special vector fields $\underline{F}$, $\int_C \underline{F}.\underline{ds}$ only depends on the endpoints. These vector fields are called <u>conservative</u>, so if C_1, C_2 are two paths with the same endpoints, then F conservative $\Rightarrow \int_{C_1} \underline{F}.\underline{ds} = \int_{C_2} \underline{F}.\underline{ds}$

<u>special case</u> C is a closed curve then

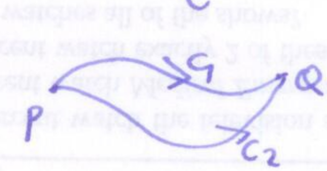
F conservative $\Rightarrow \int_C \underline{F}.\underline{ds} = 0$ <u>notation</u>: sometimes written $\oint_C \underline{F}.\underline{ds}$

<u>recall</u> if $\underline{F}$ is a gradient vector field, then $\underline{F} = \nabla f$ for some scalar function f.

<u>Thm</u> (fundamental theorem of calculus for gradient vector fields)

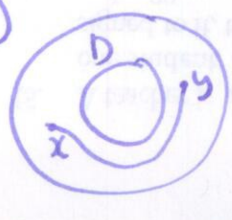
If $\underline{F} = \nabla f$ on a domain D, then for every oriented curve C in D with initial point P and final point Q, $\int_C \underline{F}.\underline{ds} = f(Q) - f(P)$

(if C is closed $P = Q$ so $\oint_C \underline{F}.\underline{ds} = 0$)



<u>Thm</u> Every conservative vector field $\underline{F}$ on an (open connected) domain D is a gradient vector field, i.e. $\underline{F} = \nabla f$ for some f

<u>Proof</u> (sketch) pick a point $\underline{x} \in D$ and define



$f(\underline{y}) = \int_C \underline{F}.\underline{ds}$ where C is any path from $\underline{x}$ to $\underline{y}$

<u>rmk</u>: this is well defined if $\underline{F}$ is conservative!

now compute $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ by choosing short horizontal/vertical paths $\square$

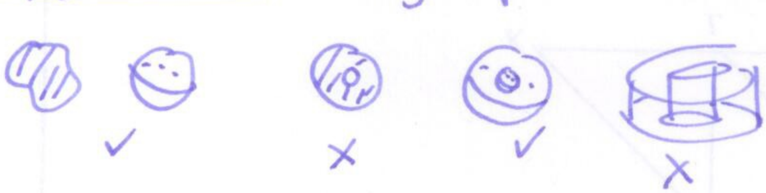
<u>Q</u>: when is $\underline{F}$ conservative? $\Leftrightarrow$ when does $\underline{F}$ have a potential function?

<u>Thm</u> Let $\underline{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field on a <u>simply connected</u> domain D. Then if the cross partials are equal

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \text{ then } \underline{F} = \nabla f \text{ for some } f.$$

$$\Leftrightarrow \boxed{\nabla \times \underline{F} = \underline{0}}$$

<u>simply connected</u>: every loop can be shrunk to a point



<u>Example</u> Vortex vector field (domain not simply connected).

$$\underline{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \quad (\text{not defined at } (0,0)!)$$

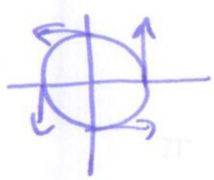
① mixed partials equal

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) &= \frac{(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) &= \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{equal!}$$

② integrate around unit circle $c(\theta) = \langle \cos\theta, \sin\theta \rangle \quad 0 \leq \theta \leq 2\pi$

$c'(\theta) = \langle -\sin\theta, \cos\theta \rangle$

$\int_C \underline{F}\, ds$



$$= \int_0^{2\pi} \underline{F}(c(\theta)) \cdot c'(\theta)\, d\theta = \int_0^{2\pi} \left\langle \frac{-\sin\theta}{1}, \frac{\cos\theta}{1} \right\rangle \cdot \langle -\sin\theta, \cos\theta \rangle\, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi \neq 0!$$