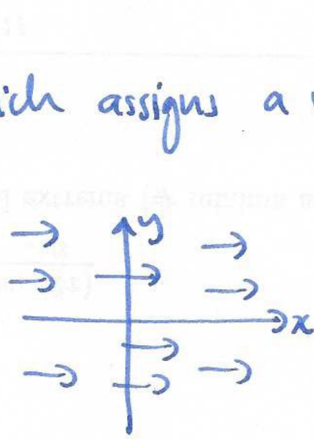
§16.1 Vector fields

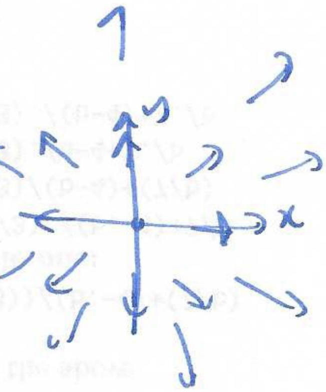
Defn A vector field is a function which assigns a vector to each point in the domain.

Examples (2d) ① $F(x,y) = \langle 1, 0 \rangle$



$F: \mathbb{R}^2 \to \mathbb{R}^2$

② $F(x,y) = \langle x, y \rangle$



③ any gradient vector field

$\underline{F} = \nabla f$ $\quad \left(\begin{array}{l} f: \mathbb{R}^2 \to \mathbb{R} \\ \nabla f: \mathbb{R}^2 \to \mathbb{R}^2 \end{array} \right).$

Defn A vector field is conservative if $F = \nabla f$ for some scalar function f, called the potential function for $\underline{F}$.

Example (3d) (inverse square field) $\underline{F}(x,y,z) = \underline{F}(\underline{x}) = \dfrac{1}{\|\underline{x}\|^3} \underline{x}$

(note $\|\underline{F}(\underline{x})\| = \dfrac{1}{\|\underline{x}\|^2}$)

claim this is conservative, with potential function $f(\underline{x}) = \dfrac{-1}{\|\underline{x}\|} = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$

check $\nabla f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle = \left\langle -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x, \ldots, \ldots \right\rangle.$

$$= \frac{\langle x, y, z \rangle}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{\underline{x}}{\|\underline{x}\|^3}.$$

Warning: not all vector fields are conservative

2d: **Thm** $\underline{F}(x,y) = \langle A(x,y), B(x,y) \rangle$ is conservative iff $\dfrac{\partial B}{\partial x} = \dfrac{\partial A}{\partial y}$

3d: **Defn** $\operatorname{curl} \underline{F} = \underline{\nabla} \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = \langle C_y - B_z, -C_x + A_x, B_x - A_y \rangle$

$\underline{F} = \langle A, B, C \rangle$

Thm $\underline{F}$ is conservative iff $\underline{\nabla} \times \underline{F} = \underline{0}$.

Finding potential functions

Example (2d) $\underline{F}(x,y) = \langle y, x \rangle = \nabla f$ for some f.

$\frac{\partial f}{\partial x} = y \Rightarrow f(x,y) = xy + c_1(y)$ but $\frac{\partial}{\partial y} c_1 = 0$

$\frac{\partial f}{\partial y} = x \Rightarrow f(x,y) = xy + c_2(x)$ $\quad \frac{\partial}{\partial x} c_2 = 0$

$\Rightarrow f(x,y) = xy + c$ constant.

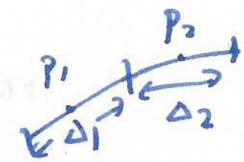
Divergence 2d: $\underline{F}(x,y) = \langle A, B \rangle$ $\quad \underline{\nabla} \cdot \underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}$ (scalar!)

3d: $\underline{F}(x,y,z) = \langle A, B, C \rangle$ $\quad \underline{\nabla} \cdot \underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$ (scalar!)

($\underline{\nabla} \cdot \underline{F} = 0$ means fluid flow is incompressible)

useful fact: $\underline{\nabla} \cdot \underline{\nabla} \times \underline{F} = 0$

§16.2 Line integrals

(parameterized) curve C :  

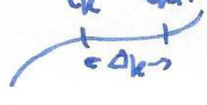
scalar line integral $\int_C f(x,y,z)\, ds$ $\quad$ ($f(x,y,z)$ scalar function $f: \mathbb{R}^3 \to \mathbb{R}$)

can define as Riemann sum $\int_C f(x,y,z)\, ds = \lim_{|\Delta s_i| \to 0} \sum_{i=1}^{n} f(P_i) \Delta_i$

Thm $\underline{c}(t)$ $a \leq t \leq b$ parameterized curve C

then $\int_C f(x,y,z)\, ds = \int_a^b f(\underline{c}(t)) \, \|\underline{c}'(t)\| \, dt$ $\leftarrow$ fact: does not depend on choice of parameterization

Note if $f(x,y,z) = 1$ then get $\int_a^b \|\underline{c}'(t)\| \, dt$ = arc length.

intuition:  length of this segment is $\int_{t_k}^{t_{k+1}} \|\underline{c}'(t)\| \, dt$