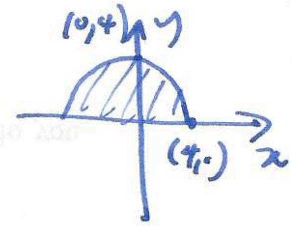
Example integrate $f(x,y) = xy$ over the upper half disc of radius 4



$$\iint_D f(x,y)\,dA \qquad \int_0^{\pi}\int_0^{4} r^2\cos\theta\sin\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{1}{2}\sin 2\theta \underbrace{\int_0^4 r^3\,dr}_{\left[\frac{1}{4}r^4\right]_0^4 = 4^3} d\theta = 32\left[-\frac{1}{2}\cos 2\theta\right]_{\pi}^{0} = 0$$

cylindrical coordinates $\iiint_R f(x,y,z)\,dV$

cartesian coords 

$\Delta V = \Delta x\Delta y\Delta z$

so $dV = dx\,dy\,dz$

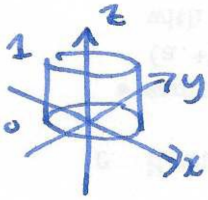
 $\Delta V \approx r\Delta r\Delta\theta\Delta z$

so $\iiint_R f(x(r,\theta), y(r,\theta), z)\; r\,dr\,d\theta\,dz$ $\quad \boxed{dV = r\,dr\,d\theta\,dz}$

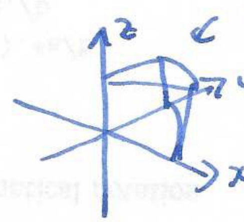
spherical coords $\boxed{dV = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$

so $\iiint_R f(x,y,z)\,dV = \iiint_R f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))\;\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

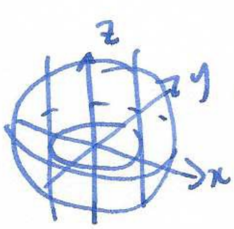
Examples



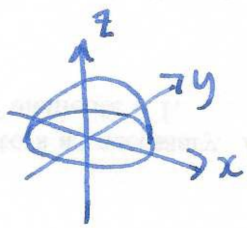
cylinder in cylindricals



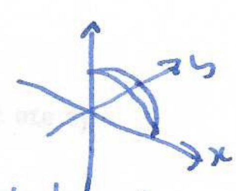
cylindrical wedge in cylindricals.



intersection of sphere with cylinder (in both).



upper half sphere in sphericals



octant sphere in sphericals.

§15.5 Change of variable

1 var $\displaystyle\int_a^b f(x)\,dx \underset{x(u)}{\longleftrightarrow} \int_c^d f(x(u))\frac{dx}{du}\,du$ $\qquad x(d)=b \quad d=x^{-1}(b)$

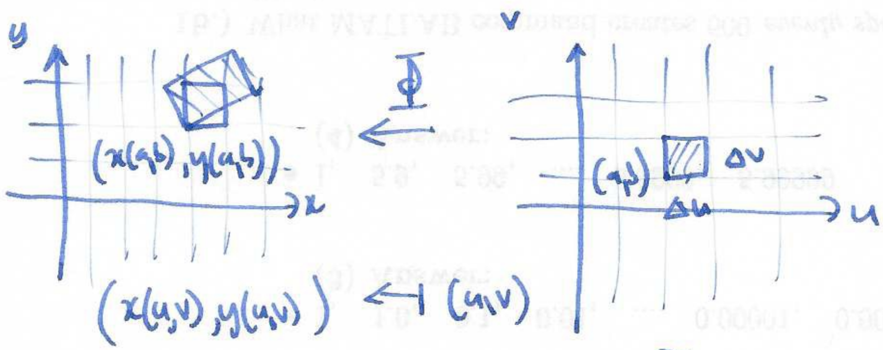
$\qquad x(c)=a \quad a=x^{-1}(a)$

2 vars $\displaystyle\iint_D f(x,y)\,dA \underset{\substack{x(u,v)\\ y(u,v)}}{\longleftrightarrow} \iint_D f(x(u,v),y(u,v))\,J\,du\,dv$

in terms of x,y $\qquad$ in terms of u,v

$$J = \left|\,\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\,\right|$$

intuition:



linear approximation: $(u,v) \overset{\Phi}{\longmapsto} (x(u,v), y(u,v))$

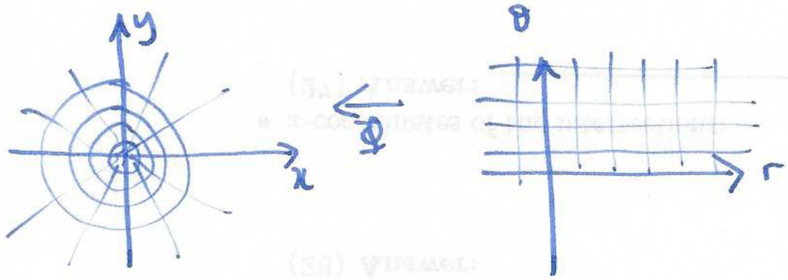
at (a,b):

$$(u,v) \mapsto (x(a,b), y(a,b)) + \left(\frac{\partial x}{\partial u}(a,b)(u-a) + \frac{\partial x}{\partial v}(a,b)(v-b),\ \frac{\partial y}{\partial u}(a,b)(u-a) + \frac{\partial y}{\partial v}(a,b)(v-b)\right) + \cdots$$

$$= (x(a,b), y(a,b)) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\begin{bmatrix} u-a \\ v-b \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \mapsto \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

change of volume is $\left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right| = J$.

Example polar coordinates

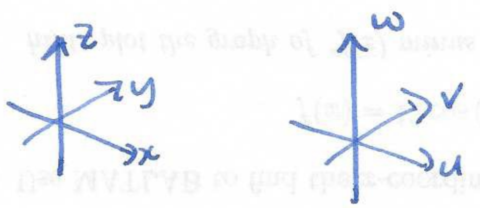


$$\Phi(r,\theta) = (\overset{x}{r\cos\theta}, \overset{y}{r\sin\theta})$$

$$J(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

so $dx\,dy = r\,dr\,d\theta$.

3 variables



$$dx\,dy\,dz = |J|\,du\,dv\,dw$$

$$J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|.$$

Example spherical coordinates

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho\cos\phi$

$$J = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\phi\sin\theta & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi \begin{vmatrix} -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \end{vmatrix} - \rho\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi \end{vmatrix}$$

$$= \rho^2\left[\cos\phi\left(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi\right) - \sin\phi\left(\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi\right)\right]$$

$$= \rho^2\sin\phi\left(\sin^2\phi + \cos^2\phi\right) = \rho^2\sin\phi.$$