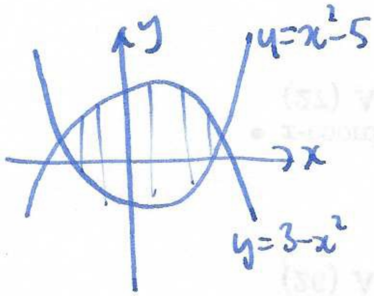
<u>Example</u> find the area of the region between $y=3-x^2$ and $y=x^2-5$



find intersections: $3-x^2 = x^2-5$

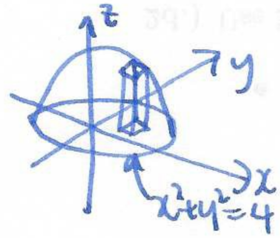
$2x^2 = 8 \quad x = \pm 2$.

$$\underbrace{\int_{-2}^{2}\int_{x^2-5}^{3-x^2} 1 \, dy\, dx} = \quad \left[y\right]_{x^2-5}^{3-x^2} = 3-x^2-(x^2-5) = 8-2x^2.$$

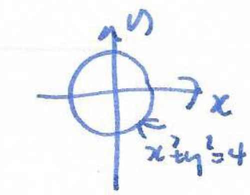
$$\int_{-2}^{2} 8-2x^2 \, dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2} = 16 - \frac{16}{3} + 16 - \frac{16}{3}$$

<u>Example</u> find the volume of the region below $z = 4-x^2-y^2$ and above the xy-plane

can do this in two ways: $\iiint_R 1 \, dV = \text{volume}$



$$\iint_D 4-x^2-y^2 \, dA$$



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} 1 \, dz\, dy\, dx \qquad \left[z\right]_0^{4-x^2-y^2} = 4-x^2-y^2.$$

$$\left[y(4-x^2)-\frac{1}{3}y^3\right]_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} = (4-x^2)^{3/2} - \frac{1}{3}(4-x^2)^{3/2} - \left(-(4-x^2)^{3/2} + \frac{1}{3}(4-x^2)^{3/2}\right)$$

$$= \frac{4}{3}(4-x^2)^{3/2}$$

$$\int_{-2}^{2} \frac{4}{3}(4-x^2)^{3/2} dx \quad \underline{\text{try}}: \ x = 2\sin\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta$$

$$\int_{-\pi/2}^{\pi/2} \frac{4}{3}\left(4-4\sin^2\theta\right)^{3/2} \cdot 2\cos\theta \, d\theta = \frac{8}{3}\int_{-\frac{\pi}{2}}^{\pi/2} 4^{3/2}\cos^3\theta \cos\theta \, d\theta \quad \text{etc...}$$

$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1}{2}\cos 2\theta + \frac{1}{2}$$

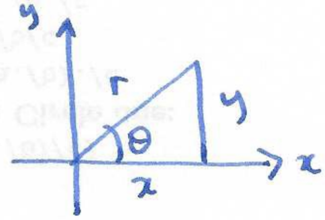
$$= \frac{64}{3}\int_{-\frac{\pi}{2}}^{\pi/2} \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right)^2 d\theta = \frac{64}{3}\int_{-2}^{2} \frac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right) d\theta$$

$$= \frac{16}{3}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{1}{2}\cos 4\theta + \frac{1}{2} + 2\cos 2\theta + 1 \, d\theta$$

$$= \frac{16}{3}\left[-\frac{1}{8}\sin 4\theta + \frac{3\theta}{2} - \sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{16}{3}\cdot\frac{3}{2}\cdot 2\cdot\frac{\pi}{2} = 8\pi.$$

§12.7 Cylindrical and spherical coordinates

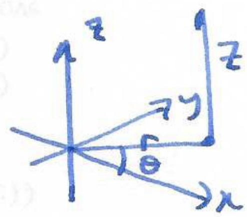
Polar coordinates:



$r^2 = x^2 + y^2$ $\qquad x = r\cos\theta$

$\tan\theta = \frac{y}{x}$ $\qquad y = r\sin\theta$

Cylindrical coordinates:

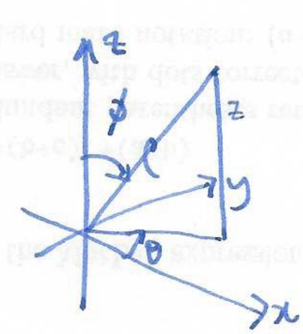


$x = r\cos\theta$ $\qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad \theta = \tan^{-1}(y/x)$

$z = z$ $\qquad z = z$

Spherical coordinates:



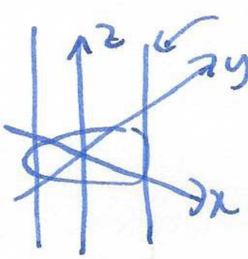
$x = \rho\cos\theta\sin\phi$ $\qquad \rho^2 = x^2+y^2+z^2$

$y = \rho\sin\theta\sin\phi$ $\qquad \theta = \tan^{-1}(y/x)$

$z = \rho\cos\phi$ $\qquad \phi = \cos^{-1}\left(\frac{z}{\rho}\right)$

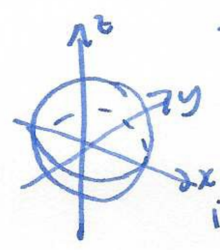
$= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$

some sets are easier to describe in these coordinates.



$x^2+y^2 \le 1 \Leftrightarrow r=1$

in cylindricals

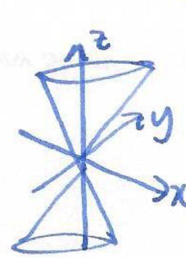


$x^2+y^2+z^2 = 1$

$\Updownarrow$

$\rho = 1$

in sphericals



$z^2 = x^2+y^2$

$\Updownarrow$

$z^2 = r^2$

in cylindricals.