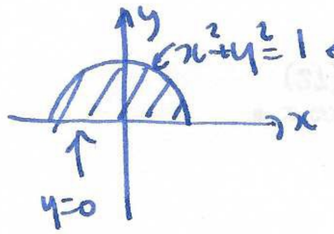
Example D = upper half disc



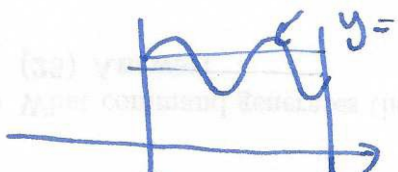
$x^2+y^2=1 \leftrightarrow y=\sqrt{1-x^2}$

$x = \pm\sqrt{1-y^2}$

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

$$\int_{0}^{1}\int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f(x,y)\,dx\,dy$$

Example D =

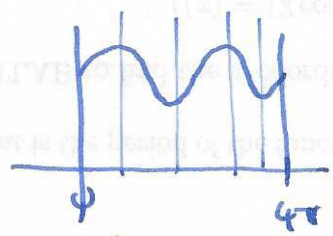


$$\int_0^{4\pi}\int_0^{2+\sin x} f(x,y)\,dy\,dx$$

$$\int_0^3\int_{*}^{*} f(x,y)\,dx\,dy$$

↑ problem! boundary not a function of y!

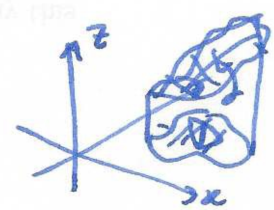
solution: subdivide region.



↑ now each subregion has boundary which is a function of y.

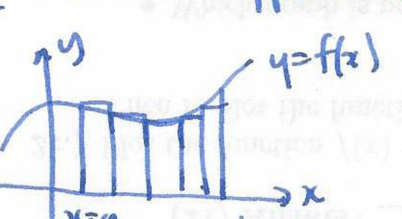
§15.2 Double integrals over general regions

notation: $\iint_D f(x,y)\,dA$ means volume under surface $z=f(x,y)$ over region D



observation: we can approximate integrals

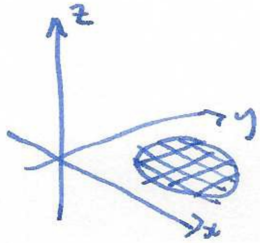
1 var



$$\int_a^b f(x)\,dx \approx \sum \underbrace{f(x_i)}_{\text{height}}\underbrace{\Delta x_i}_{\text{width}}$$

$$= \lim_{|\Delta x|\to 0} \sum f(x_i)\Delta x_i$$

2 var

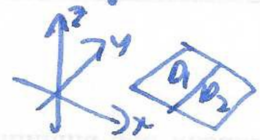


$$\iint_D f(x,y)\,dA \approx \sum \underbrace{f(x_i,y_i)}_{\text{height}}\underbrace{\Delta x\Delta y}_{\text{area of base}}$$

$$= \lim_{|\Delta x\Delta y|\to 0} \sum f(x_i,y_i)\Delta x\Delta y.$$

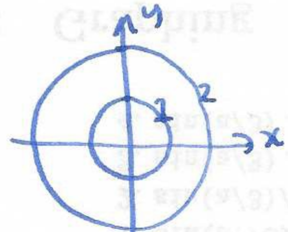
Observation · if a region D is the union of two subregions D_1, D_2 then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$



· if D is $D_1 \setminus D_2$ then $\iint_D = \iint_{D_1} - \iint_{D_2}$

Example

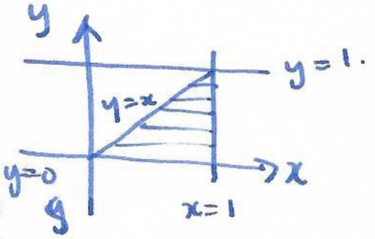


$$\iint_{\text{annulus}} = \iint_{\text{disk}} - \iint_{\circ}$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx - \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$$

also need to be able to translate limits into regions:

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$$



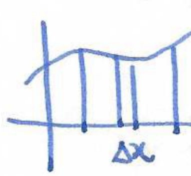
can now change order of integration

warning: to change order of integration, must draw picture.

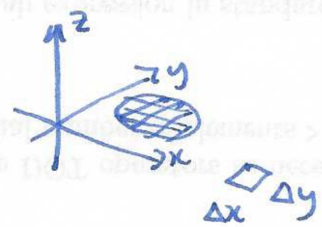
$$\int_0^1 \int_0^y \frac{\sin x}{x}\,dy\,dx \leftarrow \text{can now do integral.}$$

§15.3 Triple integrals

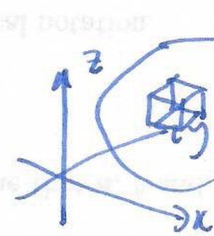
1 var

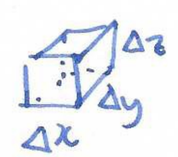


2 var



3 var





$\sum f(x_i)\,\Delta x$

$\sum f(x_i, y_j)\,\Delta x\,\Delta y$

$\sum f(x_i, y_j, z_k)\,\Delta x_i\,\Delta y_j\,\Delta z_k$

notation · $\iiint_R f(x,y,z)\,dV$

Example integrate $f(x,y,z) = x^2 e^{y+3z}$ over the box $\underset{x}{[1,2]} \times \underset{y}{[3,4]} \times \underset{z}{[0,1]}$

$$\int_0^1 \int_3^4 \underbrace{\int_1^2 x^2 e^{y+3z} \, dx}\, dy\, dz$$

$$\text{inner integral} = e^{y+3z} \int_1^2 x^2 \, dx = e^{y+3z} \left[\tfrac{1}{3} x^3 \right]_1^2 = \frac{7}{3} e^{y+3z}$$

$$\int_0^1 \underbrace{\int_3^4 \frac{7}{3} e^{y+3z} \, dy}\, dz$$

$$\text{inner integral} = \frac{7}{3} e^{3z} \int_3^4 e^y \, dy = \frac{7}{3} e^{3z} \left[e^y \right]_3^4 = \frac{7}{3}(e^4 - e^3) e^{3z}$$

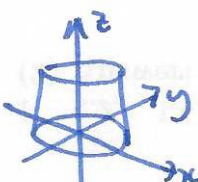
$$\int_0^1 \frac{7}{3}(e^4-e^3) e^{3z} \, dz = \frac{7}{3}(e^4 - e^3) \left[\frac{1}{3} e^{3z} \right]_0^1 = \frac{7}{9}(e^4-e^3)(e^3-1)$$

Describing regions region $R \longleftrightarrow$ limits in terms of x, y, z, in some order.

R = cylinder

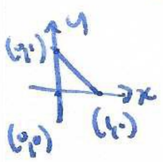
$0 \leq z \leq 1$

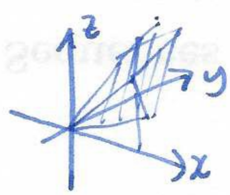
$x^2 + y^2 \leq 1$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 f(x,y,z) \, dz\, dy\, dx$$

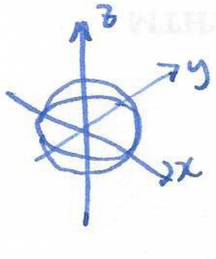
R = region between planes $z = x+y$ and $z = 3x+5y$ over the triangle





$$\int_0^1 \int_0^{1-x} \int_{x+y}^{3x+5y} f(x,y,z) \, dz\, dy\, dx$$

R = sphere $x^2+y^2+z^2 \leq 1$



$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{+\sqrt{1-x^2-y^2}} f(x,y,z) \, dz\, dy\, dx$$