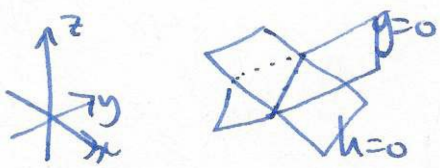
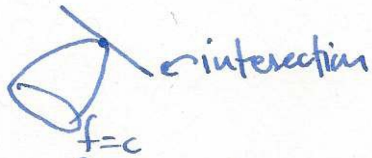
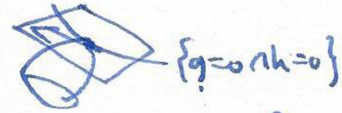
Example minimize $f(x,y,z) = x^2y^2+z^2$ subject to $x+y=2$
and $y+z=4$

solve $\nabla f = \lambda \nabla g + \mu \nabla h$

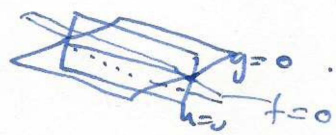
why? 

extreme points when level sets of f tangent to $\{g=0\} \cap \{h=0\}$





i.e. tangent direction to $\{g=0\}$ ~~and~~ $\{h=0\} \subset$ tangent plane to $f=c$.

span $g=0 \cap h=0 \subset$ 3rd plane 

$\Rightarrow \nabla f = \lambda \nabla g + \mu \nabla h$

$\nabla f = \langle 2xy^2, 2x^2y, 2z \rangle$

$\nabla g = \langle 1, 1, 0 \rangle$

$\nabla h = \langle 0, 1, 1 \rangle$

$$\left.\begin{aligned} 2xy^2 &= \lambda \\ 2x^2y &= \lambda + \mu \\ 2z &= \mu \end{aligned}\right\} \quad 2x^2y = 2xy^2 - 2z$$

$$\left.\begin{aligned} x+y &= 2 \\ y+z &= 4 \end{aligned}\right\} \quad \begin{aligned} y &= 2-x \\ z &= 4-y = 2+x \end{aligned}$$

$2x^2(2-x) = 2x(2-x)^2 - 2(2+x)$

↑ cubic (could use cubic formula)

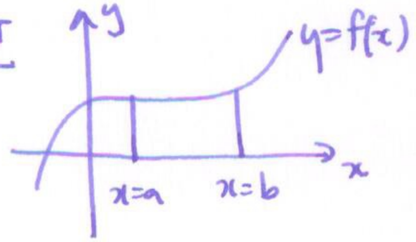
just approximate $x \simeq -0.3$

$y \simeq 2.3$

$z \simeq 1.7$.

§15.1 Integration

1 var

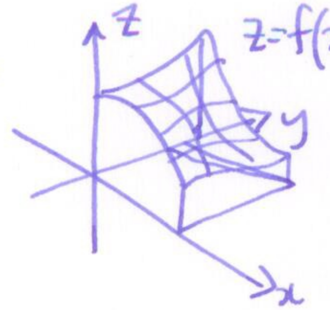


$\int_a^b f(x)\,dx$ = area under the curve between $x=a$ and $x=b$.

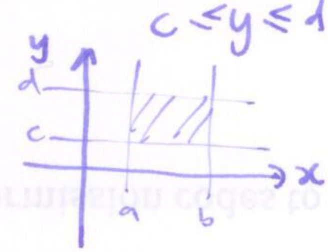
↑ compute by finding anti-derivative $F'(x)=f(x)$

then $\int_a^b f(x)\,dx = F(b)-F(a)$

2 vars



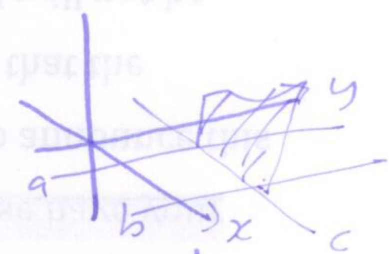
$\int_a^b \int_c^d f(x,y)\,dy\,dx$ = volume under the surface over the region $a \le x \le b$, $c \le y \le d$



↑ compute this by doing iterated integrals

recall $\frac{\partial f}{\partial x}$ = "diff wrt x keeping y constant"

$\int_a^b f(x,y)\,dx$ = "integrate wrt x keeping y constant"

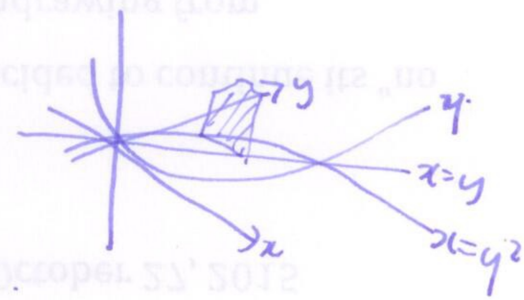


Example $\int_0^1 xy\,dx = \left[\frac{1}{2}x^2y\right]_0^1 = \frac{1}{2}y$

Note ① if you integrate wrt x there should be no x's remaining! a definite integral

② the limits can depend on y!

$$\int_{y^2}^{y^4} xy\,dx = \left[\frac{1}{2}x^2y\right]_{y^2}^{y^4} = \frac{1}{2}y^9 - \frac{1}{2}y^5$$



Warning can't have x's in the limits for $\int dx$.

Q: what does this mean? $\int_0^1 xy\,dx$ means area of slice y between $x=0$ and $x=1$.

$\int_{y^2}^{y} xy\,dx$ means area of slice y between $x=y^2$ and $x=y$.

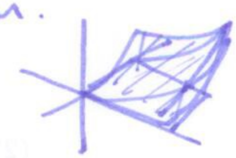
Double integrals

Example ①

$$\int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \left[\tfrac{1}{2}x^2y\right]_0^1 dy = \int_0^1 \tfrac{1}{2}y\,dy$$

describes region $0 \le y \le 1$, $0 \le x \le 1$

$$= \left[\tfrac{1}{4}y^2\right]_0^1 = \tfrac{1}{4} = \text{volume above region.}$$

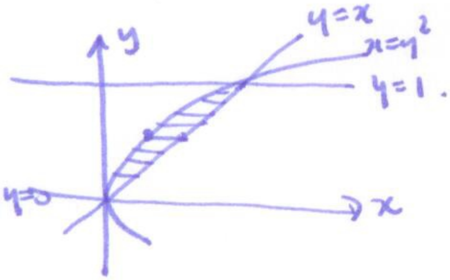


Example ②

$$\int_0^1\int_{y^2}^{y} xy\,dx\,dy = \int_0^1 \tfrac{1}{2}y^3 - \tfrac{1}{2}y^5\,dy = \left[\tfrac{1}{8}y^4 - \tfrac{1}{12}y^6\right]_0^1$$

$$= \tfrac{1}{8} - \tfrac{1}{12} = \tfrac{3-2}{24} = \tfrac{1}{24}$$

limits describe region: $0 \le y \le 1$



$y^2 \le x \le y$

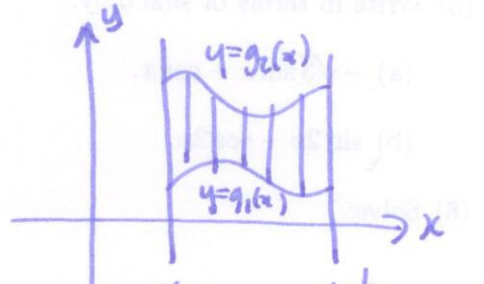
notation $\iint_D f(x,y)\,dx\,dy$ or $\iint_D f(x,y)\,dA$

↑ name of region

nice regions

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(a, b) must be constants; ($g_1(x)$, $g_2(x)$) may be functions of x



$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$