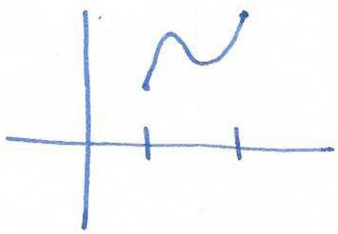
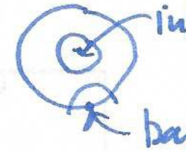
Global extrema (absolute max/min)

1d: $f: I \to \mathbb{R}$



a continuous function on a closed interval $[a,b]$ has an absolute max and an absolute min.

2d: $D \subset \mathbb{R}^2$ $f: D \to \mathbb{R}$



D is bounded if $D \subset$ disc of radius R about $(0,0)$.

$x \in D$ is an interior point if a small ball around x is contained in D

otherwise x is an boundary point

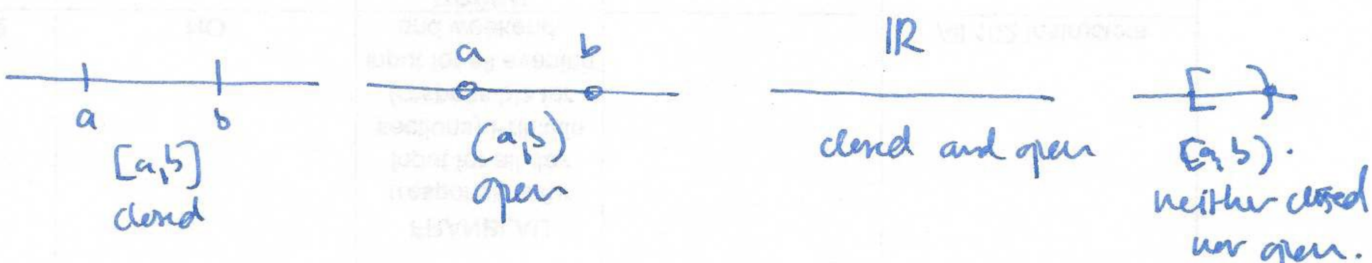
interior of D = union of all interior points

boundary of D = union of all boundary points

D is closed if D contains all its boundary points

D is open if every point is an interior point.

Examples

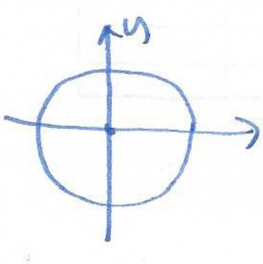


Thm $f(x,y)$ cts on $D \subset \mathbb{R}^2$ closed, bounded, then

- $f(x,y)$ has absolute max and min on D
- the extreme values occur at either critical points in the interior or on the boundary

Example find the abs max/min of $f(x,y) = xy$ on the unit disc $\{x^2+y^2=1$.

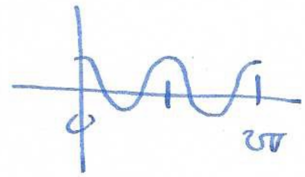
① find critical points $\left.\begin{array}{l} f_x = y \\ f_y = x \end{array}\right\}$ $(0,0)$ only critical point



② find max/min on boundary

parametrize boundary : $(\cos\theta, \sin\theta)$ $0 \le \theta \le 2\pi$

$f(\theta) = f(x(\theta), y(\theta)) = \cos\theta \sin\theta = \frac{1}{2}\sin 2\theta$

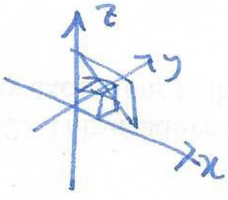


$f'(\theta) = \cos 2\theta$ $\quad f'(\theta) = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

corresponds to points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$.

Example find the max volume of a box bounded by the coordinate planes, and the plane $x+y+z=1$

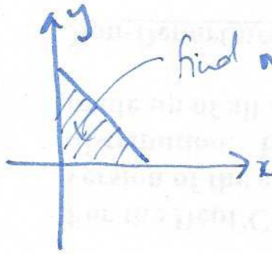


sides of box, x, y, z, with $x+y+z=1$

volume of box $V = xyz$

so sides : $x, y, z = 1-x-y$

volume : $V = xy(1-x-y)$ ← so max this on $\{(x,y) \mid x \ge 0, y \ge 0, z \ge 0 \Leftrightarrow 1-x-y \ge 0\}$ $\quad x+y \le 1$.



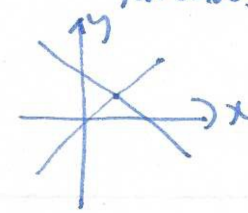
find max of V in this region

① find critical points

$V_x = y - 2xy - y^2$

$V_y = x - x^2 - 2xy$

two lines



solve $\left.\begin{matrix} V_x = 0 \\ V_y = 0 \end{matrix}\right\}$ $\begin{matrix} y - 2xy - y^2 = 0 \Rightarrow x = \frac{y-y^2}{2y} = \frac{1}{2}(1-y) \\ x - x^2 - 2xy = 0 \Rightarrow y = \frac{x-x^2}{2x} = \frac{1}{2}(1-x) \end{matrix}$

intersect at : $\left.\begin{matrix} 2x+y=1 \\ x+2y=1 \end{matrix}\right\} -3y=-1 \quad y=\frac{1}{3}, x=\frac{1}{3}$. $\quad (1/3, 1/3)$ critical point.

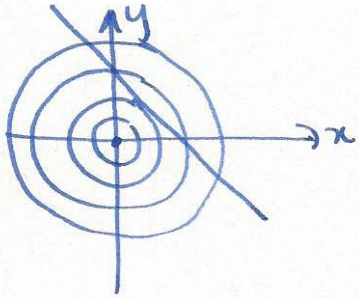
② check max on boundary : 3 edges

$x=0$, $0 \le x \le 1$	$y=0$ $\quad 0 \le y \le 1$	$(t, 1-t)$ $\quad 0 \le t \le 1 \Leftrightarrow z=0$
$V=0$	$V=0$	$V(t) = t(1-t)(1-t-(1-t))$ $= t(1-t) \cdot 0 = 0$

§14.8 Lagrange multipliers

optimization with constraint, i.e. max $f(x,y)$ subject to $g(x,y)=0$

Example ~~maximize~~ min $f(x,y)=\sqrt{x^2+y^2}$ subject to $x+y-4=0$
$g(x,y)=0$



consider level sets of $f(x,y)$, i.e. $f(x,y)=c$

key point: at the extreme values of $f(x,y)$ on $g(x,y)=0$, the level sets of f are parallel to g.

so we want to look for points where level sets of f parallel to level sets of g

i.e. solve $\nabla f = \lambda \nabla g \quad (\lambda \neq 0)$

Example $\nabla f = \left\langle \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x, \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y \right\rangle = \lambda \nabla g = \lambda \langle 1,1 \rangle$

$$\Rightarrow \left.\begin{array}{l} \dfrac{x}{\sqrt{x^2+y^2}} = \lambda \\ \dfrac{y}{\sqrt{x^2+y^2}} = \lambda \end{array}\right\} \quad \frac{①}{②}: \quad \frac{x}{y}=1 \Rightarrow x=y$$

now plug in to $g(x,y)=0$

$x+x-4=0$

$\Rightarrow x=2, y=2.$

summary want to max funcn f given $g=0$, solve $\nabla f = \lambda \nabla g$, $g=0$

Example find the point on the plane $x+2y+3z=4$ closest to the origin

i.e. min $f(x,y,z)=x^2+y^2+z^2$ subject to $g(x,y,z)=x+2y+3z-4=0$.

solve: $\nabla f = \lambda \nabla g \qquad \nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 1,2,3 \rangle$

$$\left.\begin{array}{l} 2x=\lambda \\ 2y=2\lambda \\ 2z=3\lambda \end{array}\right\} \left.\begin{array}{l} x=\lambda/2 \\ y=\lambda \\ z=3\lambda/2 \end{array}\right\} \quad \frac{\lambda}{2}+2\lambda+\frac{9\lambda}{2}=4 \quad \lambda=\frac{4}{7} \quad \text{so point is } \left\langle \frac{2}{7}, \frac{4}{7}, \frac{6}{7} \right\rangle$$