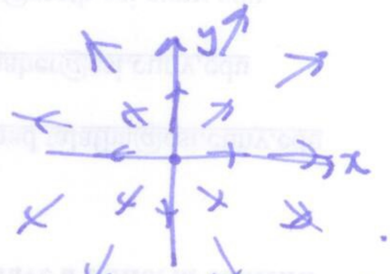
3d: $f : \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z)$ $\quad \underline{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

note $\nabla f : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \nabla f : \mathbb{R}^3 \to \mathbb{R}^3$

point $\to$ vector $\qquad$ point $\to$ vector

key facts
- the gradient vector points in the direction of fastest increase
- $\|\nabla f\|$ = fastest rate of increase.

Example $f(x,y) = x^2 + y^2$ $\qquad \nabla f = \langle 2x, 2y \rangle$



observation the gradient vector is perpendicular to the level sets / contours.

2d:  3d 

useful properties

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(cf) = c\nabla f$ $\quad c$ constant
3) product rule $\nabla(fg) = f\nabla g + g\nabla f$
4) chain rule $\nabla\big(F(f(x,y,z))\big) = F'\big(f(x,y,z)\big)\nabla f$ $\qquad F(t)$ differentiable

check this makes sense

$F \circ f \qquad \mathbb{R}^3 \xrightarrow{f} \mathbb{R} \xrightarrow{F} \mathbb{R}$

$$\nabla\big(F(f(\underline{x}))\big) = \left\langle \frac{\partial}{\partial x}\big(F(f(\underline{x}))\big), \frac{\partial}{\partial y}\big(F(f(\underline{x}))\big), \frac{\partial}{\partial z}\big(F(f(\underline{x}))\big) \right\rangle$$

$$= \left\langle F'(f(\underline{x})) \cdot \frac{\partial f}{\partial x}, F'(f(\underline{x}))\frac{\partial f}{\partial y}, F'(f(\underline{x})) \cdot \frac{\partial f}{\partial z} \right\rangle$$

$$= F'(f(\underline{x}))\, \underline{\nabla} f$$

Chain rule for paths

$f: \mathbb{R}^3 \to \mathbb{R}$ path $\underline{c}: \mathbb{R} \to \mathbb{R}^3$

$t \mapsto (x(t), y(t), z(t))$

composition $c \circ f$ $\quad \mathbb{R} \xrightarrow{c} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

Thm $\frac{d}{dt}\left(f(\underline{c}(t))\right) = \nabla f(\underline{c}(t)) \cdot \underline{c}'(t)$

in $\mathbb{R}^2$: $\frac{d}{dt}\left(f(\underline{c}(t))\right) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle x'(t), y'(t) \rangle$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example temperature T varies with location like $T(x,y,z) = \frac{1}{x^2+y^2+z^2}$

if we move along an ellipse $\frac{x-3}{25} + \frac{y^2}{16} = 1$ (foci at $(0,0)$, $(6,0)$)

parameterize ellipse: $\underline{c}(t) = (5\cos t + 3, 4\sin t, 0)$

$$T(\underline{c}(t)) = \frac{1}{(5\cos t+3)^2 + (4\sin t)^2 + (0)^2}$$

$$\frac{d}{dt}\left(T(\underline{c}(t))\right) = \nabla T \cdot \underline{c}'(t) = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

$$= \left\langle -(x^2+y^2+z^2)^{-2} \cdot 2x, -(x^2+y^2+z^2)^{-2} \cdot 2y, -(x^2+y^2+z^2)^{-2} 2z \right\rangle \cdot \langle -5\sin t, 4\cos t, 0 \rangle$$

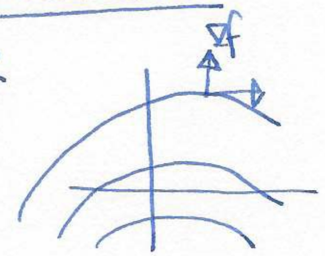
$$= \frac{-2}{(x^2+y^2+z^2)^2} \langle x, y, z \rangle \cdot \langle -5\sin t, 4\cos t, 0 \rangle$$

$$= \frac{-2}{(5\cos t+3)^2 + (4\sin t)^2} \langle 5\cos t + 3, 4\sin t, 0 \rangle \langle -5\sin t, 4\cos t, 0 \rangle$$

Directional derivatives

$f: \mathbb{R}^2 \to \mathbb{R}$

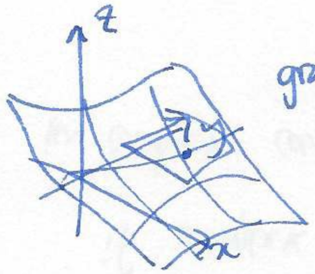
$f(x,y)$



$\underline{\nabla} f \perp$ level sets

rate of change in direction $\underline{\nabla} f = \|\underline{\nabla} f\|$

rate of change in $\perp$ direction (i.e. tangent to level set) is 0.



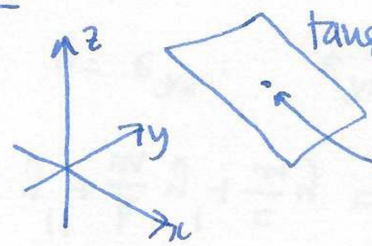
graph $z = f(x,y)$ tangent plane at a point

what about rate of change in some other direction?

<u>Thm</u> if $\underline{v}$ is (a unit) vector, the <u>directional derivative</u> $D_{\underline{v}} f$ is equal to

$$\boxed{D_{\underline{v}} f(a,b) = \underline{\nabla} f(a,b) \cdot \underline{v}}$$

<u>sketch</u>:



tangent plane has equation
$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

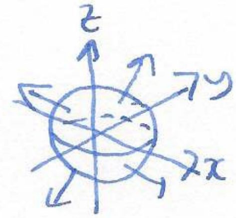
so if $\underline{v} = \langle v_1, v_2 \rangle$ then rate of change is $\frac{\partial f}{\partial x}(a,b)\, v_1 + \frac{\partial f}{\partial y}(a,b)\, v_2$

$= \underline{\nabla} f \cdot \underline{v}$

<u>Useful properties</u>

- if $\underline{v}$ is not a unit vector define $D_{\underline{v}} f(a,b) = \underline{\nabla} f \cdot \underline{v}$
- $D_{\lambda \underline{v}} f(a,b) = \lambda \underline{\nabla} f \cdot \underline{v}$
- so if $\underline{v}$ is <u>not</u> a unit vector, the directional derivative in direction $\underline{v}$ is $\frac{1}{\|\underline{v}\|} \underline{\nabla} f \cdot \underline{v}$

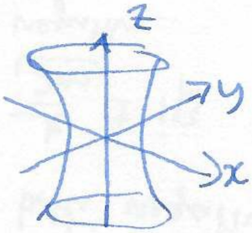
Applications



Finding normal vectors: sphere $x^2+y^2+z^2=r^2$

consider $f(x,y,z) = x^2+y^2+z^2$

then $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ is the normal vector.



Finding a tangent plane: hyperboloid $x^2+y^2=z^2+1$

find normal vector: consider $f(x,y,z) = x^2+y^2-z^2=1$

then $\nabla f = \langle 2x, 2y, -2z \rangle$,

so normal vector at $(1,1,1)$ is $\langle 2,2,-2 \rangle$

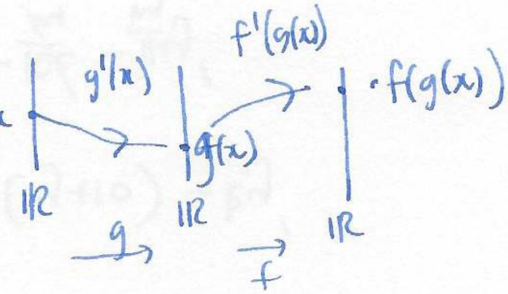
tangent plane is $\underline{n} \cdot (\underline{x}-\underline{p}) = 0$ $\quad \langle 1,1,-1 \rangle \left(\langle x,y,z \rangle - \langle 1,1,1 \rangle \right) = 0$

$$x + y - z = 1$$

§14.6 Chain rule

recall: $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$



$f: \mathbb{R} \to \mathbb{R}^2$

$\quad t \mapsto \langle x(t), y(t) \rangle$

$f'(t) = \langle x'(t), y'(t) \rangle$

$f: \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto f(x,y)$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$

In general $f: \mathbb{R}^a \to \mathbb{R}^b$ is $f(x_1, x_2, \ldots, x_a) = \left(f_1(x_1, \ldots, x_a), f_2(x_1, \ldots, x_a), \ldots, f_b(x_1, \ldots, x_a) \right)$

Example $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f(x,y) = (x^2+y^2, x^2-y^2)$

the derivative at a point is a <u>linear map</u> $Df(x_1, \ldots, x_a): \mathbb{R}^a \to \mathbb{R}^b$

given by the matrix of partial derivatives.

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$(x,y) \mapsto (ax+by, cx+dy)$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

in general $f: \mathbb{R}^a \to \mathbb{R}^b$

$$[Df]_{ij} = \left\{ \frac{\partial f_i}{\partial x_j} \right.$$