§14.3 Partial derivatives

$f: \mathbb{R}^2 \to \mathbb{R}$
$f(x,y)$

$\frac{\partial f}{\partial x}(x,y) = f_x =$ "differentiate wrt x keeping y constant"

$\frac{\partial f}{\partial y}(x,y) = f_y =$ "differentiate wrt y keeping x constant"

Example $\quad f(x,y) = x^2+y^2 \qquad \frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 2y$

$f(x,y) = xy \qquad \frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$

$f(x,y) = xye^x + x^2y \qquad \frac{\partial f}{\partial x} = ye^x + xye^x + 2xy$

$\frac{\partial f}{\partial y} = xe^x + x^2$

note $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ still functions of two variables, so we can partially differentiate again

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \qquad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

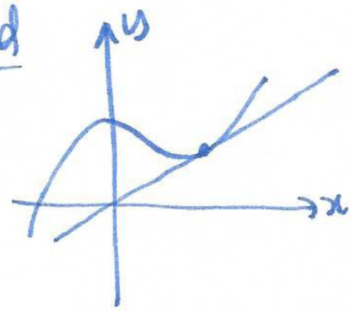
Example
$$f_{xx} = ye^x + xye^x + ye^x + 2y$$
$$\left.\begin{aligned} f_{xy} &= e^x + xe^x + 2x \\ f_{yx} &= xe^x + e^x + 2x \end{aligned}\right\} \text{equal!}$$
$$f_{yy} = 0$$

Thm If f_{xy} and f_{yx} exist and are both continuous, then $f_{xy} = f_{yx}$

"mixed partials are equal".

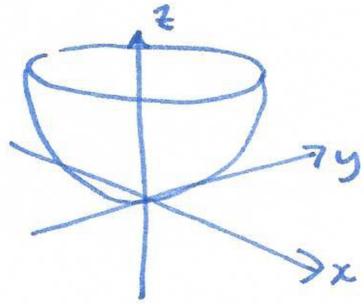
what does this mean?

<u>1d</u>



$f: \mathbb{R} \to \mathbb{R}$ $\quad \frac{\partial f}{\partial x} = \frac{df}{dx}$ = rate of change / slope of tangent line

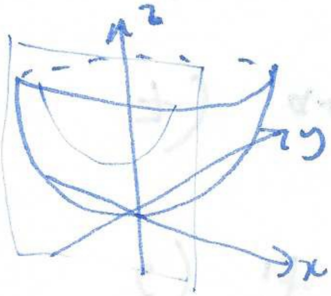
<u>2d</u> $f(x,y) = x^2 + y^2$



$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

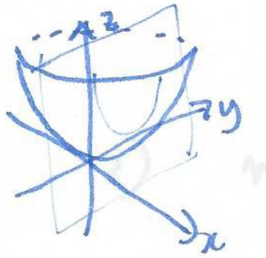
f_x = differentiate wrt x <u>keeping y fixed</u>, i.e. $y = c \Leftrightarrow$ vertical slice parallel to xz-plane



in $y=c$ f look like $f(x,c) = x^2 + c^2$
(i.e. a parabola with slope $2x$)

$$\boxed{\frac{\partial f}{\partial x} = \text{slope in } x\text{-direction}}$$

<u>keep x fixed</u>: i.e. $x = c \Leftrightarrow$ vertical slice parallel to yz-plane



in $x = c$ f looks like $f(c,y) = c^2 + y^2$
(i.e. a parabola with slope $2y$)

$$\boxed{\frac{\partial f}{\partial y} = \text{slope in } y\text{-direction}}$$

f_{xx} = "2nd derivative in x-direction"

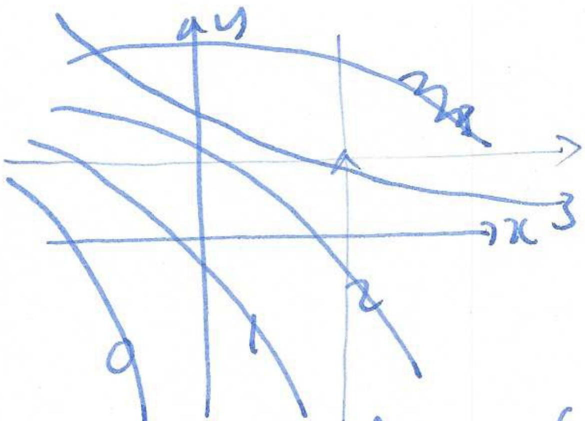
f_{yy} = "2nd derivative in y-direction"

f_{xy} = "rate of change of f_x in y-direction"

f_{yx} = "rate of change of f_y in x-direction"

<u>Warning</u> f_x, f_y etc. exist $\not\Rightarrow$ f is continuous!

interpreting contour lines | level sets of $f(x,y)$



$f_x > 0$ (numbers on contours increasing)

$f_{xx} < 0$ (contour lines getting further apart)

$f_y > 0$ (numbers on contour lines increasing)

$f_{yy} > 0$ (contour lines getting closer together)

$f_{xy} > 0$ (contour lines get closer together in x-direction as we move up in y-direction)

<u>Examples</u> ① $f(x,y) = x\sin(x+y)$

$f_x = \sin(x+y) + x\cos(x+y)$ $\qquad$ $f_y = x\cos(x+y)$

② $f(x,y) = \dfrac{ae^{xy}}{y}$ $\qquad$ $f_x = \dfrac{a\cancel{y}e^{xy}}{\cancel{y}}$ $\qquad$ $f_y = \dfrac{y\,aye^{xy} - ae^{xy}}{y^2}$

<u>Functions of 3 vars $f: \mathbb{R}^3 \to \mathbb{R}$</u>

$f(x,y,z)$ $\quad \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ = "differentiate wrt z keeping x,y fixed".

Example $f(x,y,z) = xy + yz + xyz$

$f_x = y + yz \quad f_y = x + z + xz \quad f_z = y + xy$

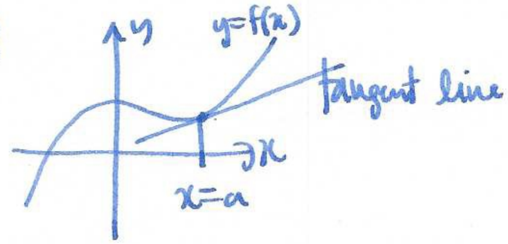
Thm "mixed partials are equal" if they exist and are continuous.

i.e. $f_{xy} = f_{yx} \quad f_{xz} = f_{zx} \quad f_{zy} = f_{yz}$.

works for $f: \mathbb{R}^n \to \mathbb{R}$. $f(x_1, \dots, x_n)$, $\frac{\partial f}{\partial x_i}$ etc...

§14.4 Differentiability, linear approximations and tangent planes

$f: \mathbb{R} \to \mathbb{R}$

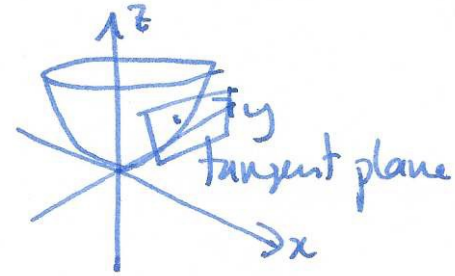


linear approximation at $x=a$ is

$$L(x) = f(a) + f'(a)(x-a)$$

tangent line is $y = L(x)$

$f: \mathbb{R}^2 \to \mathbb{R}$



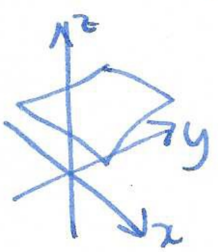
linear approximation is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

tangent plane is $z = L(x,y)$.

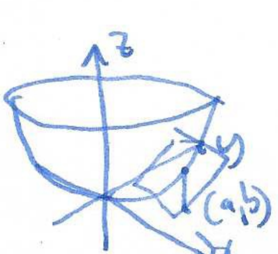
explanation

consider plane $z = cx + dy$



slope in x-direction is $\frac{\partial z}{\partial x} = c$

y-direction is $\frac{\partial z}{\partial y} = d$



at (a,b) slope in x-direction is $\frac{\partial f}{\partial x}(a,b) = f_x(a,b)$

y-direction is $\frac{\partial f}{\partial y}(a,b) = f_y(a,b)$

Example find tangent plane to $f(x,y) = x^2+y^2$ at $(1,1)$

$f(1,1)=2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$ so $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$
$= 2 + 2(x-1) + 2(y-1)$

Def^{n} $f(x,y)$ is locally linear at (a,b) if $L(x,y)$ approximates $f(x,y)$ to first order, i.e.

$$f(x,y) = L(x,y) + \underbrace{\epsilon(x,y)}_{\text{any function s.t. } \epsilon(x,y)\to 0 \text{ as } (x,y)\to(0,0)} \sqrt{(x-a)^2+(y-b)^2}$$

Problem f_x, f_y exist $\not\Rightarrow$ f locally linear.

Def^{n} $f(x,y)$ is differentiable at (a,b) if

- $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist
- $f(x,y)$ is locally linear at (a,b)

In this case the tangent plane is $z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Th^{m} If $f_x(a,b)$ and $f_y(a,b)$ exist and are continuous close to (a,b), then $f(x,y)$ is differentiable at (a,b).

Bad example $z^2 = x^2+y^2$ not differentiable at $(0,0)$

$z = \sqrt{x^2+y^2}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta \\ \frac{\partial f}{\partial y} &= \tfrac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta \end{aligned}\right\} \text{ not cts at } (0,0)$$

§14.5 Gradient

2d: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y)$

the gradient of f is $\underline{\nabla} f = \left< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right>$

$$\underline{\nabla} f(a,b) = \left< \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right>$$