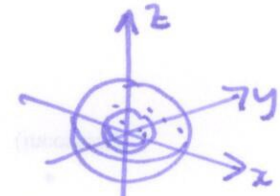
Functions of 3-vars $f(x,y,z)$

graphs: live in $\mathbb{R}^4$, hard to draw.

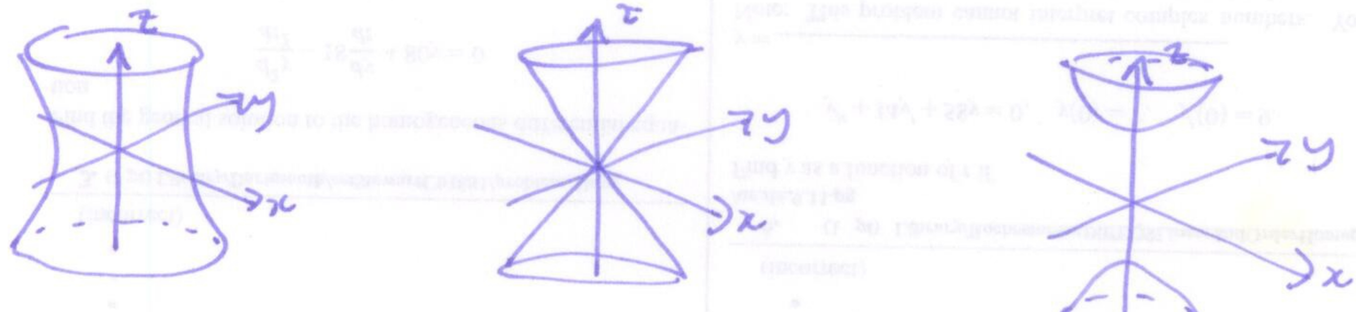
Level sets: $f(x,y,z)=c$ are surfaces in $\mathbb{R}^3$, so we can draw them.

Examples ① $f(x,y,z) = x+y+z$



② $f(x,y,z) = x^2+y^2+z^2$

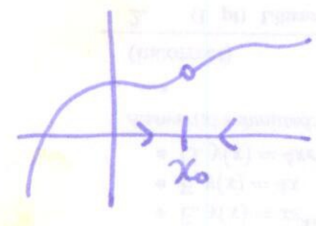
③ $f(x,y,z) = x^2+y^2-z^2$



$c>0$ $\quad$ $c=0$ $\quad$ $c<0$

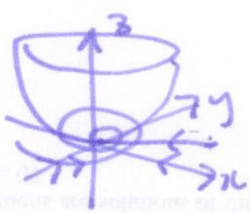
§14.2 Limits and continuity for functions of many variables

recall $y=f(x)$ $\quad \lim\limits_{x\to x_0} f(x)=L$ if $|f(x)-L|$ gets small as $|x_0-x|$ gets small.



only two directions: left and right.

2 vars $z=f(x,y)$

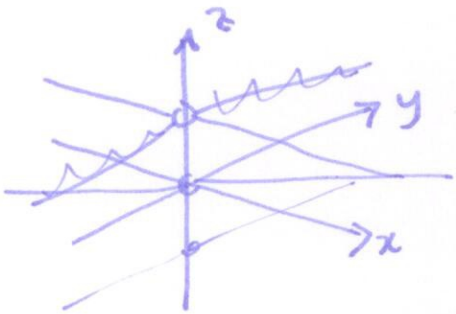


many ways to get to $(0,0)$

Def^n^ $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $|f(x,y)-L|$ gets small as $|(x,y)-(x_0,y_0)|$ gets small.

Bad example $f(x,y) = \left(\dfrac{x^2-y^2}{x^2+y^2}\right)$

Q: what happens near $(0,0)$?

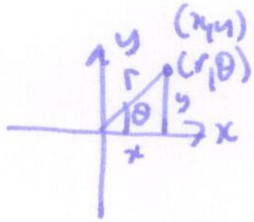


$$\lim_{(x,0)\to(0,0)} f(x,0) = \left(\frac{x^2}{x^2}\right) = 1$$

$$\lim_{(0,y)\to(0,0)} f(0,y) = \left(\frac{-y^2}{y^2}\right) = -1$$

$$\lim_{(t,t)\to(0,0)} f(t,t) = \left(\frac{0}{2t^2}\right) = 0$$

try: polar coords



$x = r\cos\theta \qquad r^2 = x^2+y^2$

$y = r\sin\theta \qquad \tan\theta = y/x$

$$\frac{x^2-y^2}{x^2+y^2} = \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = \cos^2\theta - \sin^2\theta = \cos 2\theta \qquad \lim_{(x,y)\to(0,0)} f(x,y) \text{ \underline{DNE}}.$$

Warning $\lim\limits_{(x,0)\to(x_0,y_0)} f(x,y)$ exists $\not\Rightarrow \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

Thm **Limit Laws** (assume $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim\limits_{(x,y)\to(a,b)} g(x,y)$ exists.

then:

sums: $\lim\limits_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$

constant multiple: $\lim\limits_{(x,y)\to(a,b)} cf(x,y) = c\lim\limits_{(x,y)\to(a,b)} f(x,y)$

products: $\lim\limits_{(x,y)\to(a,b)} f(x,y)\,g(x,y) = \lim\limits_{(x,y)\to(a,b)} f(x,y) \lim\limits_{(x,y)\to(a,b)} g(x,y)$

quotients: $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x,y)}{\lim\limits_{(x,y)\to(a,b)} g(x,y)}$ assuming denominator $\neq 0$.

$\underline{\text{Def}^{\underline{n}}}$ $f(x,y)$ is <u>continuous</u> at (x_0,y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

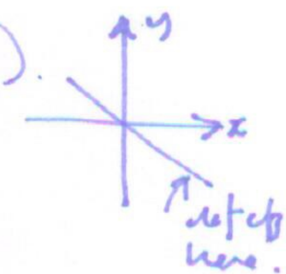
<u>Q</u>: when is $f(x,y)$ cts? A: hard to know.

however: $\underline{\text{Th}^{\underline{m}}}$ "compositions of continuous functions are continuous".

so if f, g continuous, then: fg, $f \pm g$, $\frac{f}{g}$ $(g \neq 0)$, $f \circ g$ cts.

<u>Example</u> $f(x,y) = \dfrac{x-y}{x+y}$ <u>note</u>: $\left.\begin{array}{l} f(x,y) = x \\ f(x,y) = y \end{array}\right\}$ cts. linear functions are cts.

so $x-y$ cts, $x+y$ cts, $\dfrac{x-y}{x+y}$ cts as long as $x+y \neq 0$ (i.e. $x \neq -y$).



How to show limit does not exist: $f(x,y) = \dfrac{x^2}{x^2+y^2}$ at $(0,0)$

find some directions to go to $(0,0)$ which give different answers.

$$\left.\begin{array}{l} \lim\limits_{(x,0)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \lim\limits_{(x,0)\to(0,0)} 1 = 1 \\[2em] \lim\limits_{(0,y)\to(0,0)} \dfrac{x^2}{x^2+y^2} = \lim\limits_{(0,y)\to(0,0)} 0 = 0 \end{array}\right\} \Rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2} \ \underline{\text{DNE}}.$$

similar def$^{\underline{n}}$s for functions of 3 or more variables

$\underline{\text{Th}^{\underline{m}}}$ "compositions of cts functions are cts"

so $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$ cts except at $(0,0,0)$.