<u>note</u>: $\underline{r}'(t)$ is tangent to the curve (as long as $\underline{r}'(t) \neq 0$)

$\|\underline{r}'(t)\|$ is the speed at time t.

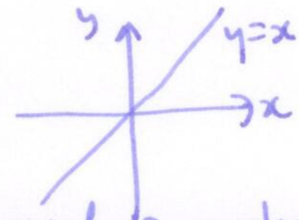
<u>Example</u> if a particle moves with position given by $\underline{r}(t) = \langle e^{2t}, t^{1/3}, \tan t \rangle$

find speed at $t=2$: $\underline{r}'(t) = \langle 2e^{2t}, \frac{1}{3}t^{-2/3}, \sec^2 t \rangle$

$\|\underline{r}'(2)\| = \|\langle 2e^4, \frac{1}{3 \cdot 2^{2/3}}, \sec^2(2) \rangle\| = \sqrt{4e^8 + \frac{1}{9 \cdot 2^{4/3}} + \sec^4(2)}$.

<u>Arc length parameterizations</u>

<u>Problem</u> parameterizations are not unique



$\underline{r}(t) = \langle t, t \rangle$

$\underline{r}(t) = \langle t^3, t^3 \rangle$

<u>special parameterizations</u>: <u>arc length</u> or <u>unit speed</u> parameterizations.

<u>Def^{n}</u> $\underline{r}(t)$ is an arc length parameterization if $\|\underline{r}'(t)\| = 1$ for all t.

(i.e. if you move along the curve with unit speed)

<u>Example</u> find an arc length parameterization for $\underline{r}(t) = \langle 2t, 1-2t, t \rangle$

① find arc length at time t:

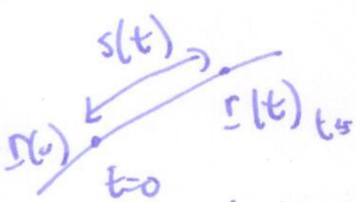
$$s(t) = \int_0^t \|\underline{r}'(u)\| \, du \qquad \underline{r}'(t) = \langle 2, -2, 1 \rangle$$

$$\|\underline{r}'(t)\| = \sqrt{4+4+1} = 3$$

so $s(t) = \int_0^t 3 \, du = \left[3u \right]_0^t = 3t$

② find the inverse function for arc length

instead of $\underline{r}(t)$, want $\underline{r}(s^{-1}(t))$



(note: arc length of $\underline{r}(s^{-1}(t))$ is $s(s^{-1}(t)) = t$)

in example $s^{-1}(t) = \frac{t}{3}$

③ write down reparameterized curve: $\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t)) = \langle \frac{2}{3}t, 1 - \frac{2}{3}t, \frac{1}{3}t \rangle$

check!

$\|\underline{r}'(t)\| = \| \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle \| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$

<u>Summary</u> $\underline{r}(t)$ arbitrary parameterization

let $s(t)$ be arc length on $[0,t]$ i.e. $s(t) = \int_0^t \|r'(u)\|\, du$

and let $s^{-1}(t)$ be the inverse function.

then the arc length parameterization is $\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t))$.

§14.1 Functions of many variables

<u>Examples</u> height above sea level $h(x,y)$
temperature $t(x,y,z)$

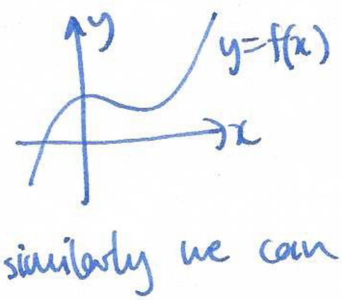
<u>Defn</u> a function of many variables is a function $f: \mathbb{R}^n \to \mathbb{R}$
domain range

or if $D \subset \mathbb{R}^n$, $f: D \to \mathbb{R}$.

<u>Examples</u> $f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \longmapsto \sqrt{9-x^2-y^2}$ Q: what is D?

$f: \mathbb{R}^3 \to \mathbb{R}$
$(x,y,z) \longmapsto x\sqrt{y} + \ln(z-1)$ Q: what is D?

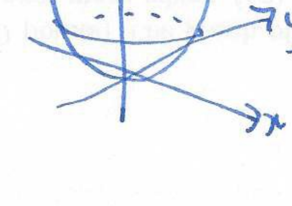
<u>Recall</u> function of one variable $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ has a <u>graph</u>



graph is collection of points (input, output) a curve in $\mathbb{R}^2$.
$(x, f(x))$

similarly we can draw the graph of a function of two variables.

$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto f(x,y)$



graph is collection of points
(input, output) $(x,y,f(x,y))$
graph is now a surface in $\mathbb{R}^3$.

Examples ① draw graph of $f(x,y) = x^2+y^2$.

traces: horizontal : solve $f(x,y)=c$ $x^2+y^2=c$ (circles of radius $\sqrt{c}$)

vertical : fix $x=c$ or $y=c$ $f(x,c) = x^2+c^2$ (parabolas)

$f(c,y) = c^2+y^2$ (parabolas)

② draw graph of $f(x,y) = x^2-y^2$

vertical traces are $f(c,y) = c^2-y^2$

$f(x,c) = x^2-c^2$

③ linear functions: $f(x,y) = ax+by+c$.

the graph of a linear function is a plane $z=ax+by+c$

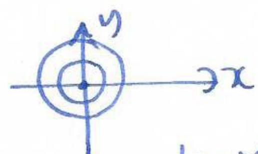
the traces are straight lines.

Contour maps

The horizontal traces are also known as contour lines or level sets

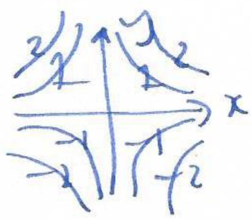
(solutions to $f(x,y)=c$) contour lines live in the domain!

Example ① $f(x,y) = \sqrt{4-x^2-y^2}$

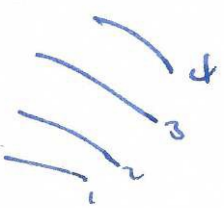


② $f(x,y) = x^2y$

$f(x,y) = c \iff x^2y=c \iff y = \frac{c}{x^2}$



On the interpretation of contours



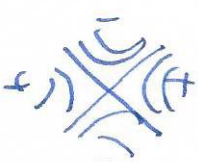
far apart $\iff$ slow change

close together $\iff$ fast change

direction of fastest change $\perp$ to contours.

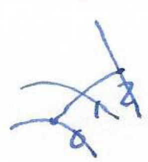
$\parallel$ contour stays same.

~~could be~~ local min/max depending on labels



saddle

average rate of change $\dfrac{\Delta f}{\|\Delta \underline{x}\|}$



$\dfrac{2}{\text{distance}}$.