**Example** ① $\underline{r}(t) = \langle t, t^2, t^3 \rangle \qquad f(t) = e^t$

$$(f(t)\underline{r}(t))' = e^t \langle t, t^2, t^3 \rangle$$

$$= e^t \langle t, t^2, t^3 \rangle + e^t \langle 1, 2t, 3t^2 \rangle$$

$$= \langle te^t + e^t,\ t^2e^t + 2te^t,\ t^3e^t + 3t^2e^t \rangle$$

② $\underline{r}(f(t)) = \underline{r}'(f(t)) \cdot f'(t)$

$$= \langle 1, 2e^t, 3e^{2t} \rangle e^t = \langle e^t, 2e^{2t}, 3e^{3t} \rangle.$$

**Thm** Product rules for dot and cross products

($\underline{r}_1(t), \underline{r}_2(t)$ differentiable)

dot product: $\frac{d}{dt}\left(\underline{r}_1(t) \cdot \underline{r}_2(t)\right) = \underline{r}_1'(t) \cdot \underline{r}_2(t) + \underline{r}_1(t) \cdot \underline{r}_2'(t)$

cross product: $\frac{d}{dt}\left(\underline{r}_1(t) \times \underline{r}_2(t)\right) = \underline{r}_1'(t) \times \underline{r}_2(t) + \underline{r}_1(t) \times \underline{r}_2'(t).$

Warning: order matters in cross product!

Proof (sketch) dot, cross products have componentwise definitions.

simple example in $\mathbb{R}^2$: $\frac{d}{dt}\left(\underline{r}_1(t) \cdot \underline{r}_2(t)\right) = \frac{d}{dt}\left(\langle x_1(t), y_1(t) \rangle \cdot \langle x_2(t), y_2(t) \rangle\right)$

$$= \frac{d}{dt}\left(x_1(t)x_2(t) + y_1(t)y_2(t)\right) = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$$

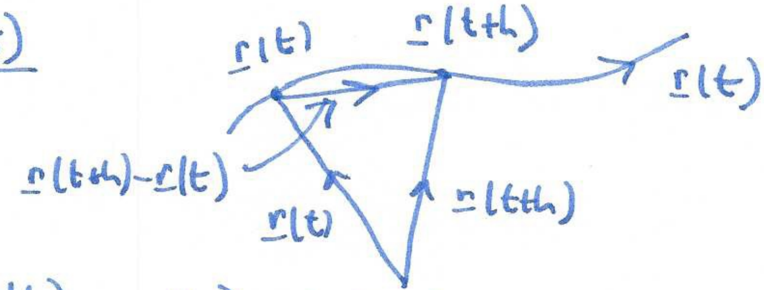
$$= x_1'x_2 + y_1'y_2 + x_1x_2' + y_1y_2' = \underline{r}_1'(t) \cdot \underline{r}_2(t) + \underline{r}_1(t) \cdot \underline{r}_2'(t) \quad \square.$$

**Example** show $\frac{d}{dt}\left(\underline{r}(t) \times \underline{r}'(t)\right) = \underline{r}(t) \times \underline{r}''(t)$

$$\frac{d}{dt}\left(\underline{r}(t) \times \underline{r}'(t)\right) = \underbrace{\underline{r}'(t) \times \underline{r}'(t)}_{=0} + \underline{r}(t) \times \underline{r}''(t)$$

The derivative is the tangent vector (velocity vector)

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$




tangent line is given by $\mathbf{L}(t) = \mathbf{r}(t_0) + t\mathbf{r}'(t_0)$
(at $\mathbf{r}(t_0)$)

Example find tangent line to the helix at $t=1$

$\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle \qquad \mathbf{r}(t) = \langle \cos 1, \sin 1, 1\rangle$

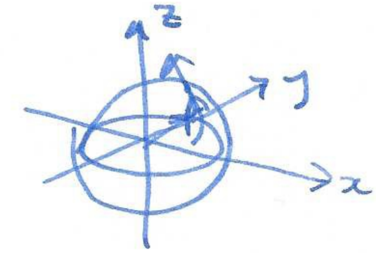
$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle \qquad \mathbf{r}'(t) = \langle -\sin 1, \cos 1, 1\rangle$

$\mathbf{L}(t) = \langle \cos 1, \sin 1, 1\rangle + t\langle -\sin 1, \cos 1, 1\rangle.$

Example show $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal if $\mathbf{r}(t)$ has unit length.

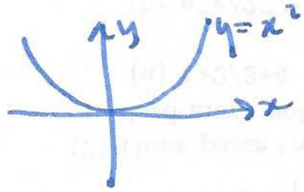
unit length $\|\mathbf{r}(t)\| = 1$
$\qquad\qquad\quad \shortparallel$
$\qquad\quad \mathbf{r}(t)\,\mathbf{r}(t)$

Explanation




$$\frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right) = \frac{d}{dt}(1) = 0$$
$\qquad \shortparallel$
$$\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \rightarrow \mathbf{r}(t), \mathbf{r}'(t) \text{ orthogonal}$$

Example




$\mathbf{r}(t) = \langle t, t^2\rangle$

$\mathbf{r}'(t) = \langle 1, 2t\rangle$ note $\|\mathbf{r}'(t)\| \neq 0$ for all $t$.

Q: what about $\mathbf{r}(t) = \langle t^3, t^6\rangle$?

Integration

Defn We define integration to be componentwise, i.e.

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \int_a^b r_3(t)\,dt \right\rangle$$

the integral exists if each of the components is integrable.

Example $\int_0^1 \langle t, t^2, \sin t \rangle \, dt = \langle \int_0^1 t \, dt, \int_0^1 t^2 dt, \int_0^1 \sin t \, dt \rangle$

$$= \langle \left[\tfrac{1}{2}t^2\right]_0^1, \left[\tfrac{1}{3}t^3\right]_0^1, \left[-\cos t\right]_0^1 \rangle$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{3}, -\cos(1)+1 \rangle.$$

Antiderivatives

Defn An antiderivative of $\underline{r}(t)$ is a function $\underline{R}(t)$ s.t. $\underline{R}'(t) = \underline{r}(t)$.

Thm Let $\underline{R}_1(t)$ and $\underline{R}_2(t)$ be antiderivatives of $\underline{r}(t)$. (i.e. $\underline{R}_1'(t) = \underline{R}_2'(t) = \underline{r}(t)$). Then $\underline{R}_1(t) = \underline{R}_2(t) + \underline{c}$, $\underline{c}$ constant vector.

General antiderivative $\int \underline{r}(t)dt = \underline{R}(t) + \underline{c}$

Fundamental theorem of calculus (vector valued version)

$\underline{r}(t)$ continuous on $[a,b]$ and $\underline{R}(t)$ an antiderivative of $\underline{r}(t)$. Then

$$\int_a^b \underline{r}(t)dt = \underline{R}(b) - \underline{R}(a)$$

Example A particle moves with velocity $\underline{r}'(t) = \langle t, \sin t \rangle$ if it starts at $\langle 1,1 \rangle$, where is it at time $t=4$?

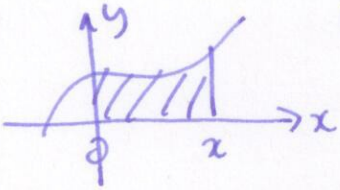
$\cancel{\int_0^4 \underline{r}(t)dt = \underline{R}(4) - \underline{R}(0)}$ $\qquad \underline{R}(t) = \langle \tfrac{1}{2}t^2, -\cos t \rangle + \underline{c}$

$t=0$: $\langle 0, -1 \rangle + \underline{c} = \langle 1,1 \rangle \Rightarrow \underline{c} = \langle 1,2 \rangle$

$\underline{R}(t) = \langle \tfrac{1}{2}t^2, -\cos t \rangle + \langle 1,2 \rangle \qquad \underline{R}(4) = \langle 8, -\cos 4 \rangle + \langle 1,2 \rangle.$

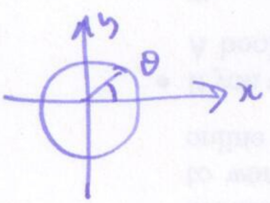
## Warning / motivation

$f:\mathbb{R}\to\mathbb{R}$

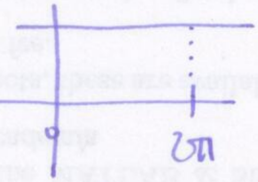


has integral $F(x) = \int_0^x f(t)\,dt$ = area under the curve

note: this doesn't work for $f$: circle $\to \mathbb{R}$

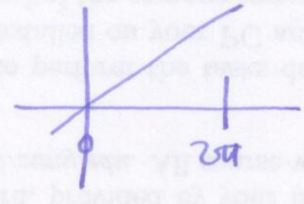


$f(\theta)$ $\quad f(0)=f(2\pi)$



$f(\theta)=1$

$f'(\theta)=0$



$F(t) = \int_0^t f(x)\,dx = t$ ?

not periodic $\quad F(0)=0$

$F(2\pi)=2\pi$ ✗

## §13.3 Arc length and speed

Recall arc length of curves in $\mathbb{R}^2$

$(x(t), y(t))$

parameterized curve $(x(t), y(t))$ $\quad t\in[a,b]$

length $L = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$

this generalizes to parameterized curves in $\mathbb{R}^3$: $\underline{r}(t) = \langle x(t), y(t), z(t)\rangle$, $t\in[a,b]$

length $L = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt = \int_a^b \|\underline{r}'(t)\|\,dt$

Example Find the arc length of $\underline{r}(t) = \langle \cos 2t, \sin 2t, 2t\rangle$ for $t\in[0,2\pi]$

$\underline{r}'(t) = \langle -2\sin 2t, 2\cos 2t, 2\rangle$

$\|\underline{r}'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 4} = \sqrt{4+4} = 2\sqrt{2}$

$L = \int_0^{2\pi} 2\sqrt{2}\,dt = 2\sqrt{2}\,\big[t\big]_0^{2\pi} = 4\pi\sqrt{2}$.