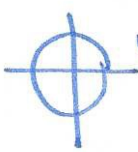
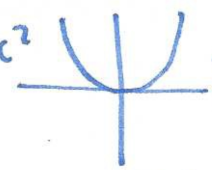
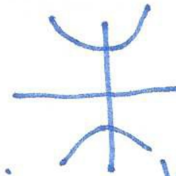
Examples $x^2+y^2=1$  unit circle. $y=x^2$  parabola  $y^2-x^2=1$ hyperbolae

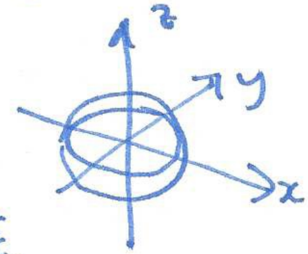
Fact: up to change of coordinates and scaling, every 2d quadric looks like this.

A quadric surface is defined by a quadratic equation in 3-variables

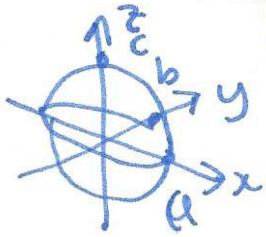
$(Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+ax+by+cz+d=0)$

↑ general quadratic in 3 vars.

Examples sphere of radius r $\quad x^2+y^2+z^2=r^2$



scaling gives ellipsoids $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$



Def: A trace is an intersection of the surface with a plane parallel to one of the coordinate planes.

Example $x^2+\frac{y^2}{4}+\frac{z^2}{9}=1$

xy-trace (set $z=0$) $\quad x^2+\frac{y^2}{4}=1$ (ellipse in xy-plane)

yz-trace (set $x=0$) $\quad \frac{y^2}{4}+\frac{z^2}{9}=1$ (ellipse in yz-plane)

trace at height

$z=1$: $\quad x^2+\frac{y^2}{4}+\frac{1}{9}=1$

$x^2+\frac{y^2}{4}=\frac{8}{9}$ (ellipse)

$z=3$: $\quad x^2+\frac{y^2}{4}+1=1$

$x^2+\frac{y^2}{4}=0$ (single point)

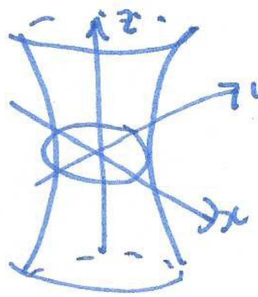
$z=4$: $\quad x^2+y^2+\frac{16}{9}=1$

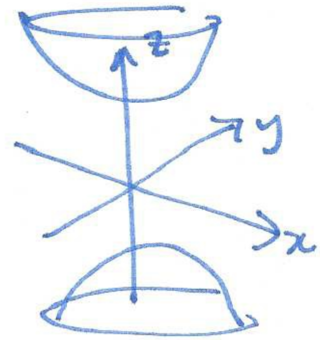
$x^2+y^2=-\frac{7}{9}$ (empty: no solutions)

hyperboloids

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=1 \qquad\qquad \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=-1$$

hyperboloid of 1 sheet — hyperboloid of two sheets

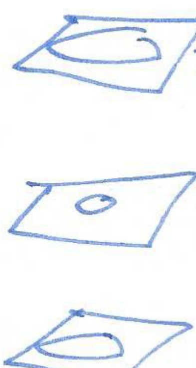




traces parallel to xy-plane : $z=z_0$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=\underbrace{1+\left(\frac{z_0}{c}\right)^2}_{}$$

always positive, so always solutions. – smallest value $z_0=0$



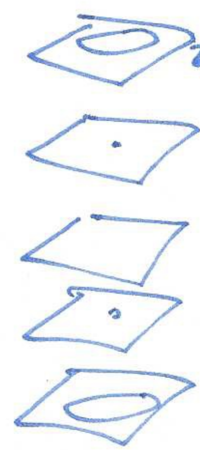
$z_0=1$

$z_0=0$

$z_0=-1$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=\underbrace{\left(\frac{z_0}{c}\right)^2-1}_{<0 \text{ if } -c<z<c}$$

so no solutions there.



$z_0>c$

$z_0=c$

$z_0=0$

$z_0=-c$

$z_0<-c$.

These hyperboloids are symmetric about z-axis (if $a=b$)

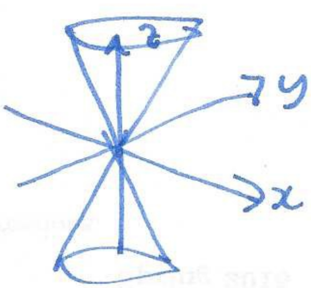
For symmetry about x-axis :

$$\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1+\left(\frac{x}{a}\right)^2 \qquad\qquad \left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=\left(\frac{x}{a}\right)^2-1$$

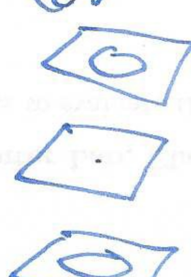
one-sheet — two sheets.

(elliptic) cones

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=\left(\frac{z}{c}\right)^2$$



traces parallel to xy-plane



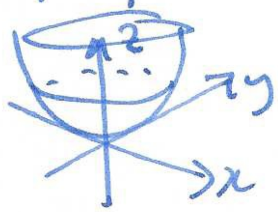
$z_0=1$

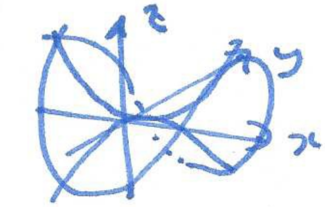
$z_0=0$

$z_0=-1$

Paraboloids $z=\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2$ $\qquad$ $z=\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2$

elliptic paraboloid $\qquad$ hyperbolic paraboloid

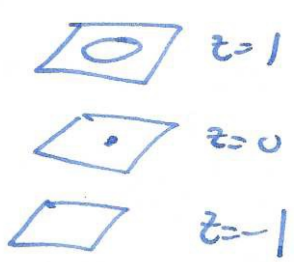




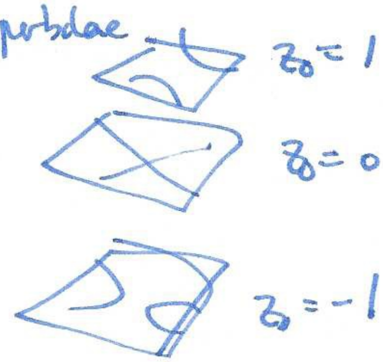
vertical traces are parabolas: $x=x_0$: $z=\left(\frac{x_0}{a}\right)^2+\left(\frac{y}{b}\right)^2$ etc.

horizontal traces: $z_0=\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2$ $\qquad$ $z_0=\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2$

ellipses

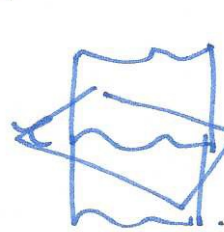


hyperbolae

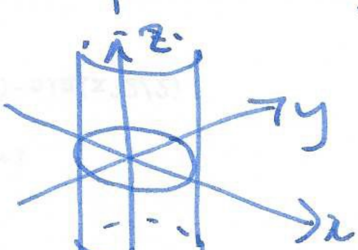


Cylinders general cylinder: $\mathcal{C}$ some curve in xy-plane

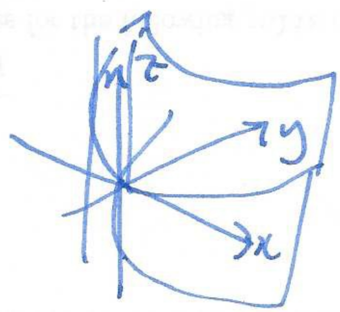
then cylinder over $\mathcal{C}$ is all points directly above or below $\mathcal{C}$



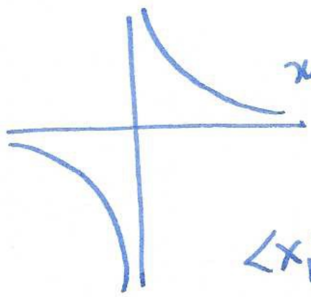
Example $x^2+y^2=r^2$



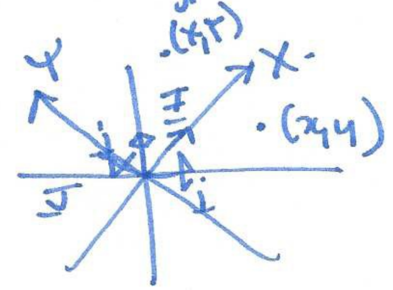
$y=x^2$



Example change of coordinates: $xy=1$ is a hyperbola.



change coords



$\langle x,y\rangle = x\underline{i}+y\underline{j}$

$$\langle X,Y\rangle = X\underline{I} + Y\underline{J}$$

$$\underline{I} = \underline{i}+\underline{j}$$
$$\underline{J} = -\underline{i}+\underline{j}$$

$$\begin{aligned}\langle X,Y\rangle &= X\underline{I} + Y\underline{J}\\ &= X(i+j)+Y(-i+j)\\ &= i(X-Y)+j(X+Y).\end{aligned}$$

so
$$x = X-Y$$
$$y = X+Y.$$

so $xy=+1 \leftrightarrow (X-Y)(X+Y)=+1$

$$X^2-Y^2=-1 \qquad Y^2 = X^2+1.$$

§3.1 Vector valued functions

real valued function of 1-variable $\quad f : \mathbb{R} \to \mathbb{R}$

$$x \mapsto f(x)$$

parameterized curves / vector valued functions

$f : \mathbb{R} \to \mathbb{R}^2 \qquad\qquad f : \mathbb{R} \to \mathbb{R}^3$

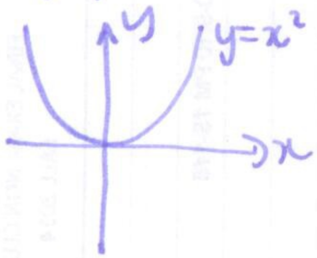
$t \mapsto \langle f(t), g(t) \rangle \qquad\qquad t \mapsto \langle f(t), g(t), h(t) \rangle$

or $\; t \mapsto f(t)\underline{i} + g(t)\underline{j} \qquad\qquad t \mapsto f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$

sometimes write $\underline{r}(t)$ to emphasize output is vector.

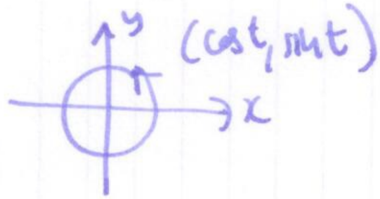
Example

① graph of a function



$x \mapsto \langle x, x^2 \rangle$ (in general $\langle x, f(x) \rangle$

② $\underline{r}(t) = \langle \cos(t), \sin(t) \rangle$



observation: $\cos^2(t) + \sin^2(t) = 1$

$$x^2 + y^2 = 1$$

warning: the same curve has many different parameterizations.

$\langle \cos t, \sin t \rangle \quad t \in \mathbb{R}$ $\qquad$ also $\quad \langle x, x \rangle$

$\langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi \qquad\qquad \langle x^3, x^3 \rangle$

$\langle \cos 2t, \sin 2t \rangle \quad 0 \le t \le \pi \qquad\qquad \langle x^2, x^2 \rangle$

Example $\; \underline{r}(t) = \langle \cos t, \sin t, t \rangle$ (helix)