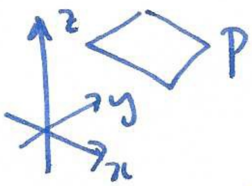
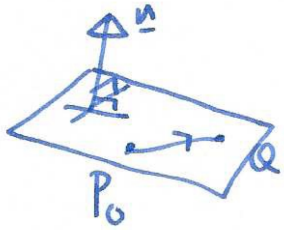
§12.5 Planes in $\mathbb{R}^3$



claim: a plane P in $\mathbb{R}^3$ is defined by a single linear equation $ax+by+cz=d$

Proof:



let P_0 be a point on P $\quad P_0 = (x_0, y_0, z_0)$

let $\underline{n}$ be the normal vector to P $\underline{n} = \langle a, b, c\rangle$

let Q be some other point on P. Then $\overrightarrow{P_0Q} \cdot \underline{n} = 0$

let $Q = (x, y, z)$, so $\overrightarrow{P_0Q} = \langle x - x_0, y - y_0, z - z_0\rangle$

$\overrightarrow{P_0Q} \cdot \underline{n} = 0$

$$\langle x - x_0, y - y_0, z - z_0\rangle \cdot \langle a, b, c\rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$$

summary $\quad ax+by+cz=d$ is equation of a plane in scalar form

$$\left.\begin{array}{l}\overrightarrow{P_0Q} \cdot \underline{n} = 0 \\ (\underline{x} - \underline{p}) \cdot \underline{n} = 0\end{array}\right\} \text{equation of a plane in vector form.}$$

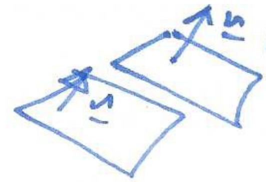
Example find equation of plane through $P_0 = (1,2,3)$ with normal vector $\underline{n} = \langle 2, 1, -1\rangle$

$$\underline{n} \cdot \overrightarrow{P_0Q} = 0 \qquad \langle 2, 1, -1\rangle \cdot \langle x-1, y-2, z-3\rangle = 0$$

$$2x - 2 + y - 2 - z + 3 = 0$$

$$2x + y - z = 1.$$

observation
- $ax+by+cz=d$ has normal vector $\underline{n} = \langle a, b, c\rangle$
- parallel planes have the same normal vectors.



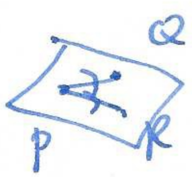
<u>Example</u>: find the plane parallel to $x - 3y + 2z = 4$ through the point $P = (6,4,2)$ $\quad \underline{n} = \langle 1,-3,2 \rangle$

$$(\underline{x} - \underline{P}) \cdot \underline{n} = 0 \qquad (x-6) - 3(y-4) + 2(z-2) = 0$$

$$x - 3y + 2z = 6 - 12 + 4 = -2$$

- Three points determine a plane

to find normal vector $\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$



<u>Example</u> $\quad P = (1,0,1) \quad Q = (2,3,1) \quad R = (-1,-1,3)$

$$\overrightarrow{PQ} = \langle 1,3,0 \rangle \qquad \overrightarrow{PR} = \langle -2,-1,2 \rangle$$

$$\underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & 0 \\ -2 & -1 & 2 \end{vmatrix} = \underline{i} \begin{vmatrix} 3 & 0 \\ -1 & 2 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}$$

$$= \langle 6,-2,-7 \rangle$$

equation: $\quad 6(x-1) - 2y - 7(z-1) = 0$

- find intersection of a point and a line

plane : $\quad 2x - 3y + 4z = 2$

line : $\quad \langle 1,2,1 \rangle + t\langle -2,1,1 \rangle \qquad \begin{aligned} x &= 1-2t \\ y &= 2+t \\ z &= 1+t \end{aligned}$

substitute in equation of plane :

$$2(1-2t) - 3(2+t) + 4(1+t) = 2$$

$$2 - 4t - 6 - 3t + 4 + 4t = 2$$

$$-3t = 2$$

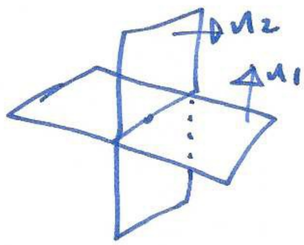
$t = -2/3$.

point is

$\langle 1,2,1 \rangle - \frac{2}{3}\langle -2,1,1 \rangle$.

check!

intersection of two planes



- direction vector for the line is perpendicular to <u>both</u> $\underline{n_1}$ and $\underline{n_2}$, so can choose it to be $\underline{n_1} \times \underline{n_2}$
- then just need to find point on ~~plane~~ line.

<u>Example</u>

$$x+y-z=2 \qquad \underline{n_1} = \langle 1,1,-1 \rangle$$
$$x+2y+z=4 \qquad \underline{n_2} = \langle 1,2,1 \rangle$$

$$\underline{n_1} \times \underline{n_2} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = \underline{i} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= \langle -3, -2, 1 \rangle$$

find a point of intersection

try: set $z=0$: $\left.\begin{array}{ll} ① & x+y=2 \\ ② & x+2y=4 \end{array}\right\}$ solve these

①-②: $-y=-2 \quad y=2 \Rightarrow x=0$ so $(0,2,0)$ works (check!)

final answer: $(\underline{x}-\underline{p}).\underline{n}=0 \qquad (\underline{x} - \langle 0,2,0 \rangle).\langle -3,-2,1 \rangle = 0$
$$-3x-2(y-2)+z=0$$
$$-3x-2y+z=-4.$$

§12.6 Quadric surfaces

~~A quadric surface is defined by a quadratic equation in 3 variables.~~

<u>recall</u> quadric curves in 2d: determined by a quadratic equation in 2 variables: $Ax^2+By^2+Cxy+Dx+Ey+F=0$ ← general quadratic in 2 vars.