§12.4 Cross product

dot product $\underline{u}.\underline{v}$ ← scalar

cross product $\underline{u}\times\underline{v}$ ← vector!

recall: A 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

eg $\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 1 = 1$

A 3x3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix}$$

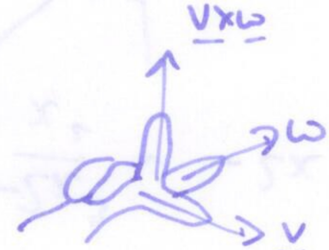
$$= 1(-1) - 2(-1) + 3(-1) = -1 + 2 - 3 = -2$$

Def$^{\underline{n}}$ (component) $\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\underline{w} = \langle w_1, w_2, w_3 \rangle$

then $\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$

geometric $\underline{v\times w}$ is the unique vector s.t.

1) $\underline{v\times w}$ is perpendicular to both $\underline{v}$ and $\underline{w}$
2) length of $\underline{v\times w}$ is $\|\underline{v}\times\underline{w}\| = \|\underline{v}\|\,\|\underline{w}\|\,|\sin\theta|$
3) $\{\underline{v}, \underline{w}, \underline{v\times w}\}$ is a right-handed triple.



Warning $\underline{v\times w} = -\underline{w\times v}$.

Example $\underline{v} = \langle 1,2,3 \rangle$ $\quad \underline{w} = \langle 1,-1,-1 \rangle$

$$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= \langle -1, 4, -3 \rangle.$$

useful properties

- $\underline{w}\times\underline{v} = -\underline{v}\times\underline{w}$ (anticommutivity)
- $\underline{v}\times\underline{v} = \underline{0}$ (zero vector!)
- $\underline{v}\times\underline{w} = \underline{0} \iff \underline{v} = \lambda\underline{w}$ for some $\lambda \in \mathbb{R}$.
- $(\lambda\underline{v})\times\underline{w} = \underline{v}\times(\lambda\underline{w}) = \lambda(\underline{v}\times\underline{w})$
- $(\underline{u}+\underline{v})\times\underline{w} = \underline{u}\times\underline{w} + \underline{v}\times\underline{w}$

$\cancel{u\times(}\ \underline{w}\times(\underline{u}+\underline{v}) = \underline{w}\times\underline{u} + \underline{w}\times\underline{v}$

special case $\quad \underline{i}\times\underline{j} = \underline{k} \qquad \underline{j}\times\underline{i} = -\underline{k} \qquad \underline{i}\times\underline{i} = \underline{0}$ etc.

$\underline{j}\times\underline{k} = \underline{i} \qquad \underline{k}\times\underline{j} = -\underline{i}$

$\underline{k}\times\underline{i} = \underline{j} \qquad \underline{i}\times\underline{k} = -\underline{j}$

alternate method of calculation

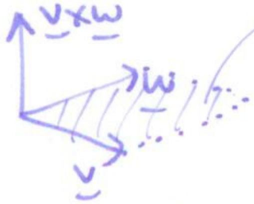
$\underline{v} = \langle 1,0,1 \rangle = \underline{i} + \underline{k} \qquad \underline{v}\times\underline{w} = (\underline{i}+\underline{k})\times(\underline{i}-\underline{j})$

$\underline{w} = \langle 1,-1,0 \rangle = \underline{i} - \underline{j}$

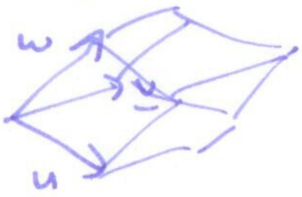
$$= \underbrace{\underline{i}\times\underline{i}}_{\underline{0}} + \underbrace{\underline{i}\times-\underline{j}}_{-\underline{k}} + \underbrace{\underline{k}\times\underline{i}}_{\underline{j}} - \underbrace{\underline{k}\times\underline{j}}_{-\underline{i}}$$

$$= +\underline{i} + \underline{j} - \underline{k} = \langle +1, 1, -1 \rangle.$$

useful facts



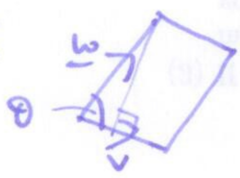
area of parallelogram determined by $\underline{v}, \underline{w}$ is $A = \|\underline{v} \times \underline{w}\|$



volume of parallelepiped determined by $\underline{u}, \underline{v}, \underline{w}$ is
$|\underline{u} \cdot (\underline{v} \times \underline{w})|$ (triple scalar product)

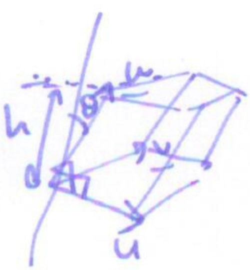
Warning $\underline{u} \cdot (\underline{v} \times \underline{w})$ makes sense $(\underline{u} \cdot \underline{v}) \times \underline{w}$ does not!

check



area is base × height

$$\|\underline{v}\| \quad \|\underline{w}\| \sin\theta = \|\underline{v} \times \underline{w}\|$$



volume is area of base × height

$$\|\underline{u} \times \underline{v}\| \quad \|w\| \cos\theta = |\underline{w} \cdot (\underline{u} \times \underline{v})| .$$

note if $\underline{u} = \langle u_1, u_2, u_3 \rangle$
$\underline{v} = \langle v_1, v_2, v_3 \rangle$
$\underline{w} = \langle w_1, w_2, w_3 \rangle$

then

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{u} \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \det \begin{pmatrix} \underline{u} \\ \underline{v} \\ \underline{w} \end{pmatrix} .$$