Math 233 Calculus 3

Joseph Maher joseph.maher@csi.cuny.edu

web page: www.math.csi.cuny.edu/~maher

office hours: M: 4:40–6:20 W 4:40–5:30 1S-222.

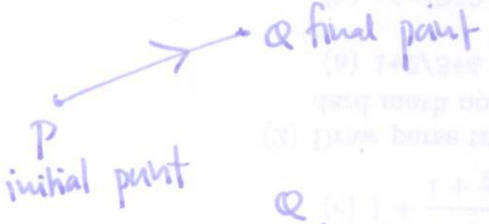
- math tutoring 1S-214
- students with disabilities

Text: Calculus (early transcendentals) Rogawski.

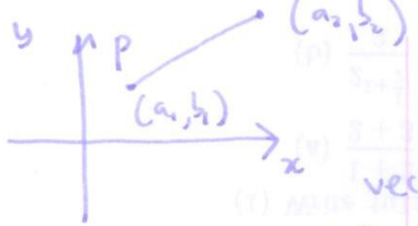
§12.1 2d vectors

scalar/number	vector
size/magnitude only	size and direction
e.g. $7, -4.3, \pi$	e.g. $\nearrow$ $\rightarrow$ length and direction
e.g. temperature, time; speed = length of velocity vector	e.g. force, velocity
notation $7, \pi \in \mathbb{R}$; $s, t \in \mathbb{R}$	$\underline{v}$, $\vec{v}$
	"length 4 in direction of +ve x-axis".

a vector $\underline{v}$ is determined by its initial and final points.



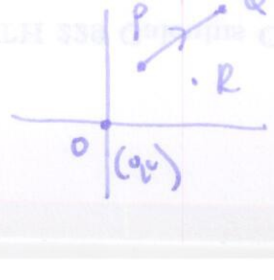
notation $\underline{v} = \vec{v} = \overrightarrow{PQ}$.



Q: how long is the vector?

A: $\|\underline{v}\| = \|\overrightarrow{PQ}\| = \sqrt{(a_2-a_1)^2 - (b_2-b_1)^2}$.

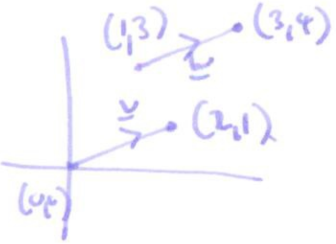
vectors vs points



every point corresponds to a special position vector from $\underline{0} = (0,0)$ to R

$\overrightarrow{OR}$

- two vectors are <u>equal</u> if they have the same length and direction



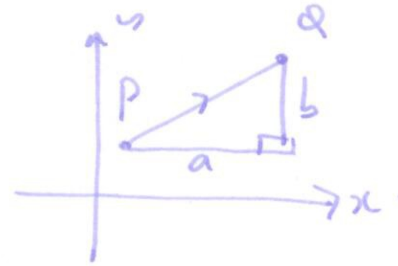
$\underset{\sim}{v} = \underset{\sim}{w}$

- two vectors are <u>parallel</u> if they have the same or opposite direction

<u>observation</u>: every vector $\underset{\sim}{v}$ is equal to a unique position vector $\underset{\sim}{v}_0$ based at the origin $O = (0,0)$

<u>Def$^{\underline{n}}$</u> the <u>components</u> of a vector $\underset{\sim}{v} = \overrightarrow{PQ}$

$P = (a_1, b_1)$ $Q = (a_2, b_2)$ are $\underset{\sim}{v} = \langle a, b \rangle$



$a = a_2 - a_1$, $b = b_2 - b_1$

x-component   y-component   <u>note</u> $\cdot \; \|\underset{\sim}{v}\| = \sqrt{a^2 + b^2}$.

- the components $\langle a, b \rangle$ determine length and directions, so two vectors are equal $\Leftrightarrow$ they have the same components.
- $\underset{\sim}{v} = \langle a, b \rangle$ does not determine an initial point.

<u>convention</u> all vectors based at O unless otherwise stated.

special vector $\underset{\sim}{O} = \langle 0, 0 \rangle$ <u>zero vector</u>.

<u>Example</u> A vector $\underset{\sim}{v}$ has length 4, lies in first quadrant, makes angle of $\pi/4$ with x-axis. find components.

$\underset{\sim}{v} = \langle a, b \rangle = \langle 4 \cos\frac{\pi}{4}, 4 \sin\frac{\pi}{4} \rangle$

<u>Vector addition and scalar multiplication</u>

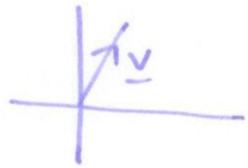
<u>scalar multiplication</u> λ number/scalar

$\underset{\sim}{v}$ vector

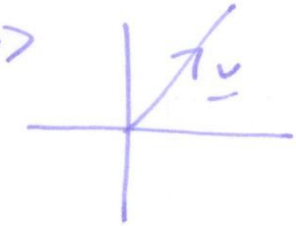
$\lambda \underline{v}$ is the vector with the same direction as $\underline{v}$ but length $|\lambda| \, \|\underline{v}\|$ if $\lambda > 0$

opposite if $\lambda < 0$

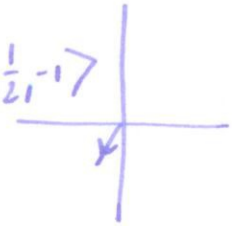
<u>Example</u> $\underline{v} = \langle 1, 2 \rangle$



$2\underline{v} = \langle 2, 4 \rangle$

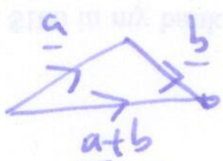


$-\frac{1}{2}\underline{v} = \langle -\frac{1}{2}, -1 \rangle$



<u>note</u>: $\underline{v}$ is parallel to $\underline{w}$ iff $\underline{v} = \lambda \underline{w}$ for some λ.

<u>vector addition</u>



$\underline{v} + \underline{w} = \underline{w} + \underline{v}$

$\underline{v} - \underline{w} = \underline{v} + (-\underline{w})$

- $\underline{v}, \underline{w}$ vectors, translate $\underline{w}$ so that beginning of $\underline{w}$ is end of $\underline{v}$, then $\underline{v} + \underline{w}$ is vector from beginning of $\underline{v}$ to end of $\underline{w}$.

<u>vector operations in components</u>

if $\underline{v} = \langle a, b \rangle$ $\underline{w} = \langle c, d \rangle$

then
- $\lambda \underline{v} = \langle \lambda a, \lambda b \rangle$
- $\underline{v} + \underline{w} = \langle a, b \rangle + \langle c, d \rangle = \langle a+c, b+d \rangle$
- $\underline{v} - \underline{w} = \langle a-c, b-d \rangle$
- $\underline{v} + \underline{0} = \langle a, b \rangle + \langle 0, 0 \rangle = \langle a, b \rangle$

<u>important</u> $\underline{v} - \underline{v} = \underline{0} = \langle 0, 0 \rangle$ zero <u>vector</u> not not zero number.

useful properties $\underline{u}, \underline{v}, \underline{w}$ vectors λ scalar

- $\underline{u}+\underline{v} = \underline{v}+\underline{u}$ (commutative)
- $\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$ (associative)
- $\lambda(\underline{v}+\underline{w}) = \lambda\underline{v}+\lambda\underline{w}$ (distributive)
- length $\|\lambda\underline{v}\| = |\lambda|\,\|\underline{v}\|$

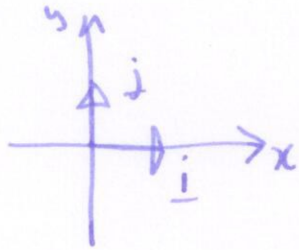
important: $\lambda+\underline{v}$ does not make sense!

unit vectors : a vector of length 1 is called a unit vector

if $\underline{v}\neq\underline{0}$ then $\hat{\underline{v}} = \hat{e}_v = \frac{1}{\|\underline{v}\|}\underline{v}$ is a unit vector.

check : $\left\|\frac{1}{\|\underline{v}\|}\underline{v}\right\| = \left|\frac{1}{\|\underline{v}\|}\right|\,\|\underline{v}\| = \frac{\|\underline{v}\|}{\|\underline{v}\|} = 1$.

special vectors



$\underline{i} = \langle 1,0\rangle$ unit vector in x-direction

$\underline{j} = \langle 0,1\rangle$ unit vector in y-direction

$\underline{i}, \underline{j}$ called standard basis vectors

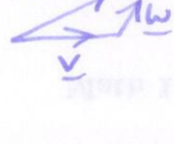
linear combinations

$\underline{v}, \underline{w}$ vectors, r, s scalars, then $r\underline{v}+s\underline{w}$ is a linear combination of $\underline{v}$ and $\underline{w}$.

Every vector can be written as a linear combination of $\underline{i}$ and $\underline{j}$.

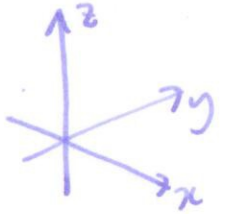
$\underline{v} = \langle a,b\rangle = a\langle 1,0\rangle + b\langle 0,1\rangle = a\underline{i}+b\underline{j}$.

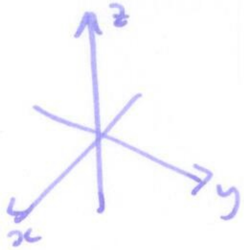
triangle inequality



$\|\underline{v}+\underline{w}\| \le \|\underline{v}\|+\|\underline{w}\|$

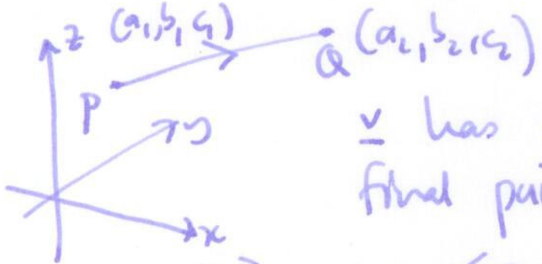
§12.2 Vectors in 3d





right hand rule

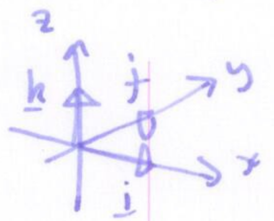
only use right handed coordinate systems.



$\underline{v}$ has length and direction; determined by initial and final points

$\vec{PQ} = \underline{v} = \langle \underbrace{a_2-a_1}_{a}, \underbrace{b_2-b_1}_{b}, \underbrace{c_2-c_1}_{c} \rangle \qquad \|\underline{v}\| = \sqrt{a^2+b^2+c^2}$

- vector addition $\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\qquad \underline{v}+\underline{w} = \langle v_1+w_1, v_2+w_2, v_3+w_3 \rangle$
  $\underline{w} = \langle w_1, w_2, w_3 \rangle$
- scalar multiplication $\qquad \lambda\underline{v} = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$
- standard basis vectors



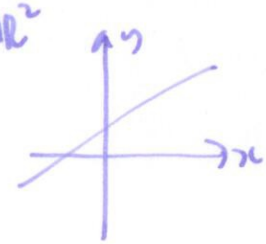
$\underline{i} = \langle 1,0,0 \rangle$
$\underline{j} = \langle 0,1,0 \rangle$
$\underline{k} = \langle 0,0,1 \rangle$

- every vector is a linear combination of standard basis vectors

$\underline{v} = \langle a,b,c \rangle = a\underline{i} + b\underline{j} + c\underline{k} = a\langle 1,0,0 \rangle + b\langle 0,1,0 \rangle + c\langle 0,0,1 \rangle$
$= \langle a,b,c \rangle .$

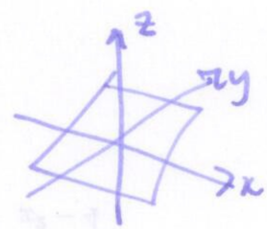
Equations of lines in $\mathbb{R}^3$

$\mathbb{R}^2$



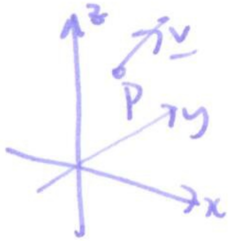
$y = mx + b$
$ax + by = c$
equation of line

$\mathbb{R}^3$



$ax + by + cz = d$
equation of a plane!

for lines in $\mathbb{R}^3$ we can use a parametric equation $\quad \underline{r}(t)$



a line L is determined by: a point P on L

a direction $\underline{v}$

any point on L is then of the form $\underline{r}(t) = \overrightarrow{OP} + t\underline{v}$

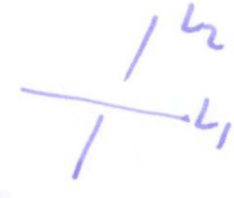
t is called the <u>parameter</u> if $\overrightarrow{OP} = \langle a, b, c \rangle$

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

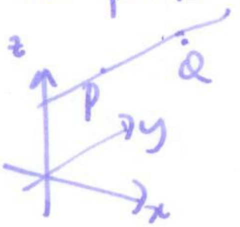
then $\underline{r}(t) = \langle a, b, c \rangle + t\langle v_1, v_2, v_3 \rangle = \langle a + tv_1, b + tv_2, c + tv_3 \rangle$

$\mathbb{R}^2$: any two lines either intersect or are parallel

$\mathbb{R}^3$: two lines may intersect, or be parallel, or neither (skew)



Two points determine a line:



$P = (p_1, p_2, p_3)$ $\qquad \underline{v} = \overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$

$Q = (q_1, q_2, q_3)$

so $\underline{r}(t) = \langle p_1, p_2, p_3 \rangle + t\langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$

$= \overrightarrow{OP} + t\overrightarrow{PQ} = \overrightarrow{OP} + t\underline{v}$.

<u>note</u> same line has many different parametrizations.

<u>useful fact</u>: $\underline{r}(t) = \overrightarrow{OP} + t\underline{v}$ $\qquad t = 0$ get $\overrightarrow{OP}$

$t = 1$ get $\overrightarrow{OQ}$

$t = \frac{1}{2}$ get midpoint of PQ.

<u>§12.3 Dot products and angles</u>

<u>Defn</u> geometric: $\underline{v} \cdot \underline{w} = \|\underline{v}\| \|\underline{w}\| \cos\theta$

components: $\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

<u>useful properties</u>:

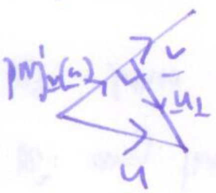
- $\underline{0} \cdot \underline{v} = 0$
- $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$ (commutativity)
- $\lambda\underline{v} \cdot \underline{w} = \underline{v} \cdot \lambda\underline{w} = \lambda(\underline{v} \cdot \underline{w})$ (scalar associativity dist).

- $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$ (distributivity)
- length $\underline{v} \cdot \underline{v} = \|\underline{v}\|^2$

<u>Defn</u> $\underline{u}, \underline{v}$ are <u>perpendicular</u> / <u>orthogonal</u> if $\underline{u} \cdot \underline{v} = 0$ $\underline{u} \perp \underline{v}$

<u>Projections</u>

can write $\underline{u}$ as a sum of a vector <u>parallel</u> to $\underline{v}$ and (projection) a vector <u>perpendicular</u> to $\underline{v}$.



$$\underline{u} = \mathrm{proj}_{\underline{v}}(\underline{u}) + \underline{u}_{\perp}$$

<u>Q</u>: how to find these? $\mathrm{proj}_{\underline{v}}(\underline{u}) = \lambda \underline{v}$ for some λ.

$$\underline{u}_{\perp} = \underline{u} - \mathrm{proj}_{\underline{v}}(\underline{u})$$

use $\underline{u}_{\perp} \cdot \underline{v} = 0$:

$$(\underline{u} - \mathrm{proj}_{\underline{v}}(\underline{u})) \cdot \underline{v} = 0$$

$$\underline{u} \cdot \underline{v} - \lambda \cancel{\underline{u}}\underline{v} \cdot \underline{v} = 0 \Rightarrow \lambda = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}}$$

so $\mathrm{proj}_{\underline{v}}(\underline{u}) = \underbrace{\frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}}}\,\underline{v}$ $\underline{u}_{\perp} = \underline{u} - \mathrm{proj}_{\underline{v}}(\underline{u})$

this is called the <u>component</u> of $\underline{u}$ in the direction $\underline{v}$.

<u>Example</u> $\underline{u} = \langle 1, 2, 3 \rangle$ $\underline{v} = \langle 1, 1, 1 \rangle$

$$\mathrm{proj}_{\underline{v}}(\underline{u}) = \frac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v} = \frac{1+2+3}{1+1+1} \langle 1,1,1 \rangle$$

$$= 2 \langle 1,1,1 \rangle = \langle 2,2,2 \rangle.$$

$$\underline{u}_{\perp} = \langle 1,2,3 \rangle - \langle 2,2,2 \rangle = \langle -1, 0, 1 \rangle$$

check $\underline{u}_{\perp} \cdot \underline{v} = 0$.