Math 233 Calculus 3 Fall 15 Midterm 2b

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find the direction of fastest increase for the function $f(x, y, z) = e^{-yz} + \ln(x + z)$ at the point $(1, 1, 2)$.

$$\nabla f = \left\langle \frac{1}{x+z}, -ze^{-yz}, -ye^{-yz} + \frac{1}{x+z} \right\rangle$$

$$\nabla f(1,1,2) = \left\langle \frac{1}{3}, -\frac{2}{e^2}, -\frac{1}{e^2} + \frac{1}{3} \right\rangle$$

(2) (10 points) Find and classify the critical points for the function $f(x, y) = xye^x$.

$$\frac{\partial f}{\partial x} = ye^x + xye^x \qquad \Rightarrow y = 0$$

$$\frac{\partial f}{\partial y} = xe^x \qquad = 0 \quad \Rightarrow x = 0$$

critical point $(0, c)$

$$f_{xx} = ye^x + ye^x + xye^x$$

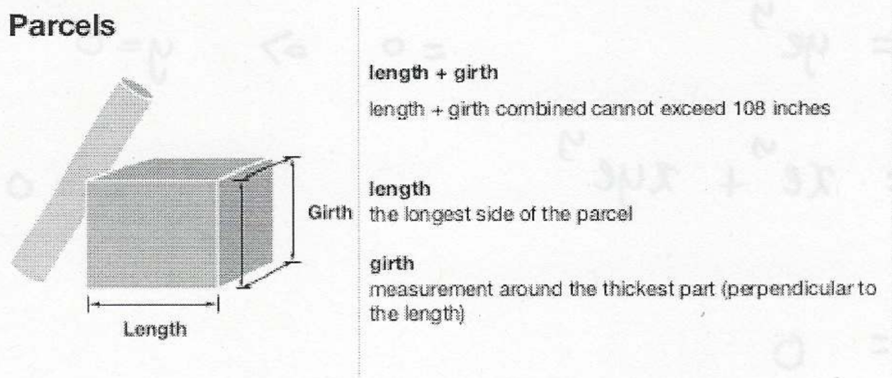
$$f_{yy} = 0$$

$$f_{xy} = e^x + xe^x$$

$$D = f_{xx} f_{yy} - (f_{xy})^2 \qquad D(0, 0) \qquad 0 \cdot 0 - (1)^2 = -1 < 0$$

saddle

(3) (10 points) what is the volume of the largest rectangular box you can send by USPS standard parcel mail?



max $V = xyz$

subject to $x + 2y + 2z = 108$

$g(x,y,z) = 108$

solve $\nabla V = \lambda \nabla g$

$g = 108$

$\nabla V = \langle yz, xz, xy \rangle$

$\nabla g = \langle 1, 2, 2 \rangle$

$yz = \lambda$

$xz = 2\lambda$

$xy = 2\lambda$

$x + 2y + 2z = 108$

$\frac{y}{x} = \frac{1}{2}$ $\quad y = \frac{1}{2}x$

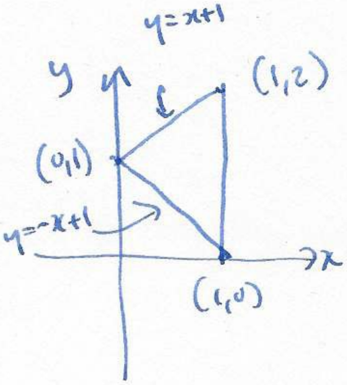
$\frac{z}{y} = 1$ $\quad y = z = \frac{1}{2}x$

$x + x + x = 108$ $\quad x = \frac{108}{3} = 36$

$y = 18 = z$

max $V = 36 \times 18^2 \text{ in}^3$.

(4) (10 points) Find the integral of the function $f(x, y) = x - y$ over the triangle with vertices $(1, 0), (0, 1)$ and $(1, 2)$.



$$\int_0^1 \int_{-x+1}^{x+1} x - y \, dy \, dx$$

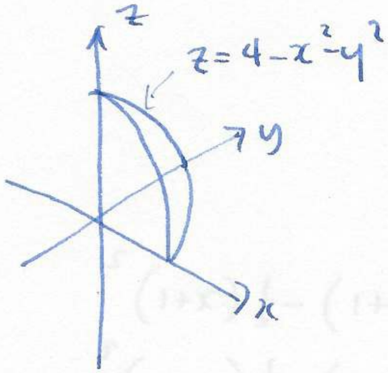
$$\left[xy - \frac{1}{2}y^2\right]_{-x+1}^{x+1} = x(x+1) - \frac{1}{2}(x+1)^2 - x(-x+1) + \frac{1}{2}(-x+1)^2$$

$$= x^2 + \cancel{x} - \frac{1}{2}x^2 - x - \cancel{\frac{1}{2}}$$
$$x^2 - \cancel{x} + \cancel{\frac{1}{2}x^2} - x + \cancel{\frac{1}{2}}.$$

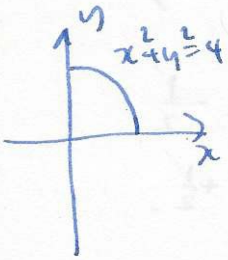
$$= 2x^2 - 2x$$

$$\int_0^1 2x^2 - 2x \, dx = \left[\frac{2}{3}x^3 - x^2\right]_0^1 = \frac{2}{3} - 1 = -1/3.$$

(5) (10 points) Write down limits for the integral of the function $f(x, y, z) = xyz$ over the region inside the positive octant $x \geq 0, y \geq 0, z \geq 0$, but underneath the surface $z = 4 - x^2 - y^2$. Do not evaluate the integral.

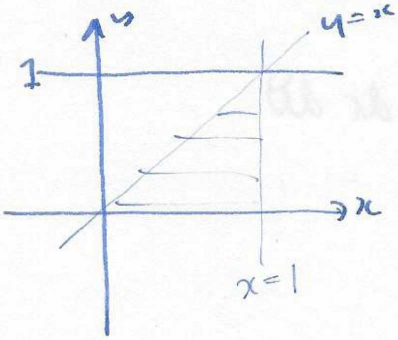


$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} xyz \quad dz\, dy\, dx$$



(6) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_y^1 \cos(x^2)\ dx\ dy$$

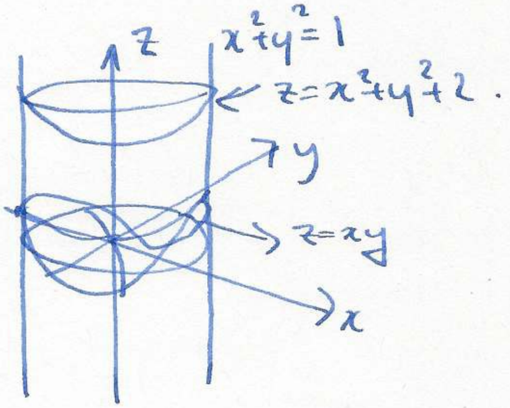


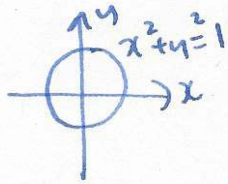
$$\int_0^1 \int_0^x \cos(x^2)\, dy\, dx$$

$$\Big[y\cos(x^2) \Big]_0^x = x\cos(x^2).$$

$$\int_0^1 x\cos(x^2)\, dx = \left[\tfrac{1}{2}\sin(x^2) \right]_0^1 = \tfrac{1}{2}\sin(1)$$

(7) (10 points) Find the volume of the region inside the cylinder $x^2 + y^2 = 1$ between the surfaces $z = x^2 + y^2 + 2$ and $z = xy$





$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1\int_{r^2\sin\theta\cos\theta}^{r^2+2} 1.\ r\,dz\,dr\,d\theta$$

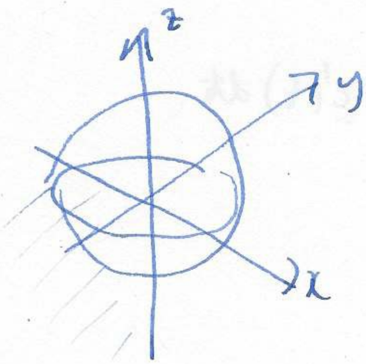
$$\Big[zr\Big]_{r^2\sin\theta\cos\theta}^{r^2+2} = r^3+2r - r^3\sin\theta\cos\theta$$

$$\left[\frac{1}{4}r^4 + r^2 - \frac{1}{4}r^4\sin\theta\cos\theta\right]_0^1 = \frac{1}{4}+1-\frac{1}{4}\sin\theta\cos\theta$$

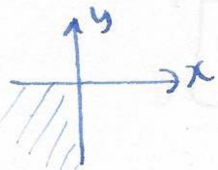
$$= \frac{5}{4} - \frac{1}{8}\sin 2\theta$$

$$\left[\frac{5}{4}\theta + \frac{1}{16}\cos 2\theta\right]_0^{2\pi} = \frac{10\pi}{4} = \frac{5}{2}\pi .$$

(8) (10 points) Write down limits for the integral of the function $f(x, y, z) = x^2 + y^2 + z^2$ in the octant $x \leq 0, y \leq 0, z \leq 0$ iside the sphere of radius 5. Do not evaluate this integral.



$$\int_{\pi}^{\frac{3\pi}{2}} \int_{\pi/2}^{\pi} \int_{0}^{5} \rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



(9) (10 points) Integrate the vector field $\mathbf{F} = \langle y, -x, z\rangle$ along the straight line from $\langle 1, 0, -2\rangle$ to $\langle 3, 1, 2\rangle$.

$\underline{c}(t) = \langle 1, 0, -2\rangle + \langle 2, 1, 4\rangle t$

$\underline{c}'(t) = \langle 2, 1, 4\rangle$

$\int_0^1 \underline{F}(\underline{c}(t)) \cdot \underline{c}'(t)\, dt$

$= \int_0^1 \langle t, -1-2t, -2+4t\rangle \cdot \langle 2, 1, 4\rangle\, dt$

$= \int_0^1 2t - 1 - 2t - 8 + 16t\, dt = \int_0^1 16t - 9\, dt = \left[\frac{16}{2}t^2 - 9t\right]_0^1$

$= \frac{\cancel{17}}{2} - 9 = -\frac{1}{2} - 1$

(10) (10 points) Show that the vector field $\mathbf{F} = \langle y, x + ze^{yz}, ye^{yz} \rangle$ is conservative, and find the potential function. Use this to evaluate

$$\int_C \mathbf{F}.\mathbf{ds},$$

where C is the curve from $(0,0,0)$ to $(1,2,5)$ formed by the intersection of the surfaces $z = x^2 + y^2$ and $z = 3x + y$.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x+ze^{yz} & ye^{yz} \end{vmatrix} = \langle e^{yz} + yze^{yz} - \cancel{y}ze^{yz} - yze^{yz}, -0+0, 1-1 \rangle$$

$$= \underline{0} \Rightarrow \text{conservative, and } F = \nabla f.$$

$$\left.\begin{array}{lll} \frac{\partial f}{\partial x} = y & & f = xy + c_1(y,z) \\ \frac{\partial f}{\partial y} = x+ze^{yz} & & f = xy + e^{yz} + c_2(x,z) \\ \frac{\partial f}{\partial z} = ye^{yz} & & f = e^{yz} + c_3(x,y) \end{array}\right\} f = xy + e^{yz} + c$$

$$\int_C \underline{F}.\underline{ds} = f(1,2,5) - f(0,0,0) = 2 + e^{10} - 1 = 1 + e^{10}.$$