Math 233 Calculus 3 Fall 15 Midterm 2a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, and a 3×5 index card of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 1	
Overall	

(1) (10 points) Find the direction of fastest increase for the function $f(x, y, z) = e^{-xz} + \ln(y + z)$ at the point $(1, 2, 1)$.

$$\nabla f = \left\langle -ze^{-xz}, \frac{1}{y+z}, -xe^{-xz} + \frac{1}{y+z} \right\rangle$$

$$\nabla f(1,2,1) = \left\langle -\frac{1}{e}, \frac{1}{3}, -\frac{1}{e} + \frac{1}{3} \right\rangle$$

(2) (10 points) Find and classify the critical points for the function $f(x, y) = xye^y$.

$\frac{\partial f}{\partial x} = ye^y = 0 \Rightarrow y = 0$

$\frac{\partial f}{\partial y} = xe^y + xye^y$ $x = 0$

$(0,0)$ critical point

$f_{xx} = 0$

$f_{xy} = e^y + ye^y$

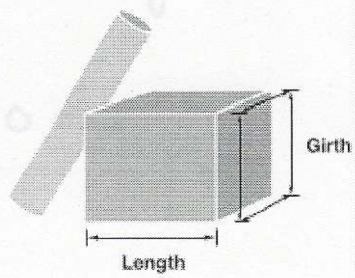
$f_{yy} = xe^y + xe^y + xye^y$

$D = f_{xx} f_{yy} - (f_{xy})^2$ $D(0,0) = 0 \cdot 0 - (1)^2 = -1 < 0$

$\Rightarrow$ saddle.

(3) (10 points) what is the volume of the largest rectangular box you can send by USPS standard parcel mail?

Parcels



length + girth
length + girth combined cannot exceed 108 inches

length
the longest side of the parcel

girth
measurement around the thickest part (perpendicular to the length)

max $V = xyz$

subject to $x + 2y + 2z = 108$

$g(x,y,z) = 108$

solve $\nabla V = \lambda \nabla g$

$g = 108$

$\nabla V = \langle yz, xz, xy \rangle$

$\nabla g = \langle 1, 2, 2 \rangle$

$yz = \lambda$

$xz = 2\lambda$

$xy = 2\lambda$

$x + 2y + 2z = 108$

$\frac{y}{x} = \frac{1}{2}$ $\quad y = \frac{1}{2}x$

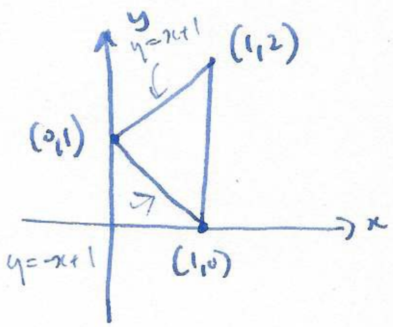
$\frac{z}{y} = 1$ $\quad y = z = \frac{1}{2}x$

$x + x + x = 108$ $\quad x = \frac{108}{3} = 36$

$y = 18 = z$

max $V = 36 \times 18^2$ in^3.

(4) (10 points) Find the integral of the function $f(x, y) = x + y$ over the triangle with vertices $(1, 0), (0, 1)$ and $(1, 2)$.



$$\int_0^1 \int_{-x+1}^{x+1} x+y \quad dy\, dx$$

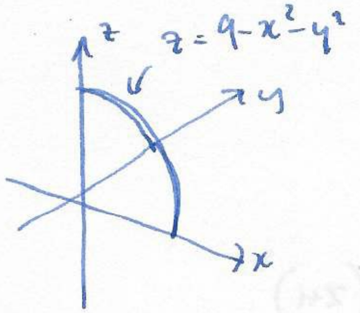
$$\left[xy + \frac{1}{2}y^2\right]_{-x+1}^{x+1} = x(x+1) + \frac{1}{2}(x+1)^2 - x(-x+1) - \frac{1}{2}(-x+1)^2$$

$$= x^2 + \cancel{x} + \cancel{\tfrac{1}{2}x^2} + x + \cancel{\tfrac{1}{2}}$$
$$x^2 - \cancel{x} - \cancel{\tfrac{1}{2}x^2} + x - \cancel{\tfrac{1}{2}}$$

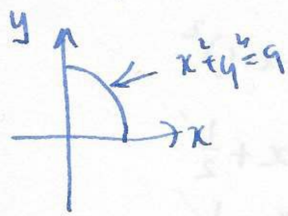
$$= 2x^2 + 2x$$

$$\int_0^1 2x^2 + 2x \, dx = \left[\frac{2}{3}x^3 + x^2\right]_0^1 = \frac{2}{3} + 1 = \frac{5}{3}$$

(5) (10 points) Write down limits for the integral of the function $f(x, y, z) = xyz$ over the region inside the positive octant $x \geq 0, y \geq 0, z \geq 0$, but underneath the surface $z = 9 - x^2 - y^2$. Do not evaluate the integral.

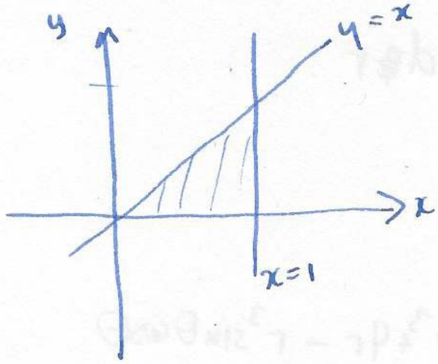


$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} xyz \, dz \, dy \, dx$$



(6) (10 points) Evaluate the following integral by changing the order of integration.

$$\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$$



$$\int_0^1 \int_0^x \sin(x^2)\, dy\, dx$$

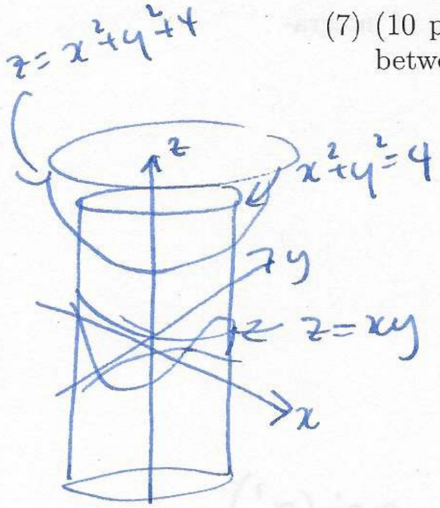
$$\Big[y \sin(x^2) \Big]_0^x = x \sin(x^2).$$

$$\int_0^1 x \sin(x^2)\, dx = \Big[-\tfrac{1}{2} \cos(x^2) \Big]_0^1$$

$$= -\tfrac{1}{2}\cos(1) + \tfrac{1}{2}.$$

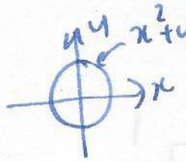
(7) (10 points) Find the volume of the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = x^2 + y^2 + 4$ and $z = xy$

$x = r\cos\theta$
$y = r\sin\theta$



$$\int_0^2 \int_0^{2\pi} \int_{r^2\sin\theta\cos\theta}^{r^2+4} 1\, r\, dz\, d\theta\, dr$$

$x^2+y^2=4 \Leftrightarrow r^2=4$
$r=2$



$$\Big[zr\Big]_{r^2\sin\theta\cos\theta}^{r^2+4} = r^3+4r - r^3\sin\theta\cos\theta$$

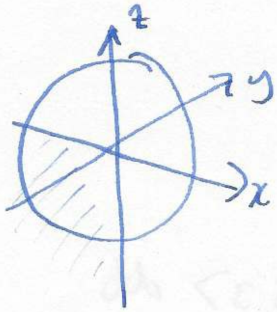
$$\int_0^{2\pi} r^3 + 4r - r^3\sin\theta\cos\theta\, d\theta$$

$$\Big[(r^3+4r)\,\theta - r^3\tfrac{1}{2}\sin^2\theta\Big]_0^{2\pi} = 2\pi(r^3+4r)$$

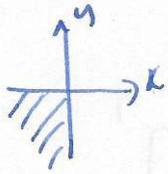
$$\int_0^2 2\pi(r^3+4r)\,dr = 2\pi\Big[\tfrac{1}{4}r^4 + 2r^2\Big]_0^2 = 2\pi(4+8)$$

$$= 24\pi.$$

(8) (10 points) Write down limits for the integral of the function $f(x, y, z) = x^2 + y^2 + z^2$ in the octant $x \leq 0, y \leq 0, z \leq 0$ iside the sphere of radius 3. Do not evaluate this integral.



$$\int_{\pi}^{3\pi/2} \int_{\pi/2}^{\pi} \int_0^3 \rho^2 \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



(9) (10 points) Integrate the vector field $\mathbf{F} = \langle y, -x, z\rangle$ along the straight line from $\langle 1, -2, 0\rangle$ to $\langle 2, 3, 3\rangle$.

$$\underline{c}(t) = (1,-2,0) + (1,5,3)t \qquad 0 \le t \le 1$$

$$\underline{c}'(t) = (1,5,3)$$

$$\int_0^1 \underline{F}(\underline{c}(t)) \cdot \underline{c}'(t)\, dt = \int_0^1 \langle -2+5t, -1-t, 3t\rangle \cdot \langle 1,5,3\rangle\, dt$$

$$= \int_0^1 -2+5t-5-5t+9t\, dt = \int_0^1 -7+9t\, dt = \left[-7t+\frac{9}{2}t^2\right]_0^1$$

$$= -7+\frac{9}{2} = -\frac{5}{2}$$

(10) (10 points) Show that the vector field $\mathbf{F} = \langle ze^{xz}, z, y + xe^{xz} \rangle$ is conservative, and find the potential function. Use this to evaluate

$$\int_C \mathbf{F}.\mathbf{ds},$$

where C is the curve from $(0,0,0)$ to $(1,2,5)$ formed by the intersection of the surfaces $z = x^2 + y^2$ and $z = 3x + y$.

$$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{xz} & z & y + xe^{xz} \end{vmatrix} = \langle 1 - 1, e^{xz} + xze^{xz} - e^{xz} - xze^{xz}, 0 - 0 \rangle = \underline{0}$$

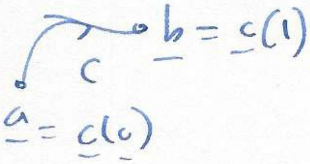
$$\frac{\partial f}{\partial x} = ze^{xz} \qquad f = e^{xz} + c_1(y,z)$$

$$\frac{\partial f}{\partial y} = z \qquad f = yz + c_2(x,z)$$

$$\frac{\partial f}{\partial z} = y + xe^{xz} \qquad f = yz + c_3(x,y)$$

$$f = yz + e^{xz} + c$$

$$\int_C \underline{F} . \underline{ds} = f(\underline{b}) - f(\underline{a})$$



$$= f(1,2,5) - f(0,0,0) = 10 + e^5 - 0 - 1 = 9 + e^5$$