Sample midterm2 Solutions

Q1 $z = f(x,y) = 4x^2 - 2xy - y^2$ $\quad \nabla f = \langle 8x - 2y, -2x - 2y \rangle$

$\nabla f(1,2) = \langle 4, -6 \rangle \leftarrow$ direction of fastest increase.

Q2 $T(x,y,z) = \dfrac{10^5}{x^2+y^2+z^2}$ $\qquad \nabla T = \dfrac{-10^5}{(x^2+y^2+z^2)^2} \langle 2x, 2y, 2z \rangle$

$\underline{r}(t) = \langle 2t, t^2 - 16, t \rangle$ $\quad \underline{r}'(t) = \langle 2, 2t, 1 \rangle$

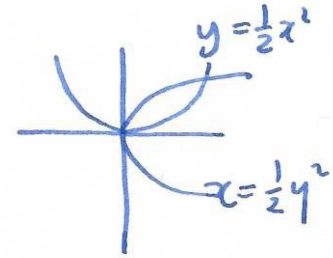
$(T(\underline{r}(t)))' = \underline{\nabla} T(\underline{r}(t)) \cdot \underline{r}'(t) = \dfrac{-10^5}{(4^2+12^2+2^2)^2} \langle 8, -24, 4 \rangle \cdot \langle 2, 4, 1 \rangle$

$\underline{r}(2) = \langle 4, -12, 2 \rangle$

$\underline{r}'(2) = \langle 2, 4, 1 \rangle$ $\qquad = \dfrac{-10^5 \cdot 76}{164^2} \approx -282.6$

Q3 a) $f(x,y) = x^3 - 6xy + y^3$

$\underline{\nabla} f = \langle 3x^2 - 6y, -6x + 3y^2 \rangle$ $\qquad \underline{\nabla} f = \underline{0}$ $\quad \begin{matrix} 3x^2 - 6y = 0 \\ -6x + 3y^2 = 0 \end{matrix}$



$x = \frac{1}{2}y^2,\ y = \frac{1}{2}x^2 \Rightarrow x = \frac{1}{8}x^4 \quad x(8 - x^3) = 0 \quad x = 0, 2.$

$(0,0)$ $(2,2)$ critical points

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = -6$

$D(0,0) = 0 - (-6)^2 = -36 < 0$ saddle.

$D(2,2) = 12^2 - (-6)^2 = 108 > 0 \quad f_{xx} > 0 \Rightarrow$ minimum.

b) $f(x,y) = 3xe^y - e^x$

$\nabla f = \langle 3e^y - e^x, 3xe^y \rangle$ $\quad \underline{\nabla} f = 0:$ $\quad 3e^y - e^x = 0$

$3xe^y = 0 \Rightarrow x = 0$

$\Rightarrow y = \ln(1/3)$

critical points $(0, \ln(1/3))$

$f_{xx} = -e^x$, $f_{yy} = 3xe^y$ $\quad f_{xy} = 3e^y$

$D(0, \ln(1/3)) = 0 - 3(1/3) = -1 < 0$ saddle.

c) $f(x,y) = 2x\ln(x+y)$ $\quad \nabla f = \langle 2\ln(x+y) + \frac{2x}{x+y}, \frac{2x}{x+y} \rangle$

$\nabla f = \underline{0}$ $\quad 2\ln(x+y) + \frac{2x}{x+y} = 0$

$\frac{2x}{x+y} = 0 \Rightarrow x=0 \Rightarrow 2\ln(y) = 0 \Rightarrow y = 1.$

critical point $(0,1)$

$f_{xx} = \frac{2}{x+y} + \frac{(x+y).2 - 2x}{(x+y)^2}$ $\quad f_{yy} = \frac{-2x}{(x+y)^2}$ $\quad f_{xy} = \frac{2}{x+y} - \frac{2x}{(x+y)^2}$

$D(0,1) = (2+2)(0) - (2-2) = 0$ no information ☹

<u>Q4</u> $f(x,y) = 2x^2 - 4y^2$ $\quad \nabla f = \langle 4x, -8y \rangle$ critical point $(0,0)$

(no critical points in interior).

on boundary: $y=0,\ 0 \le x \le 1$ $\quad f(x,0) = 2x^2$ $\quad f'(x) = 4x \Rightarrow x=0\ (0,0)$

$y=1,\ 0 \le x \le 1$ $\quad f(x,1) = 2x^2 - 4$ $\quad f'(x) = 4x \Rightarrow x=0\ (0,1)$

$x=0,\ 0 \le y \le 1$ $\quad f(0,y) = -4y^2$ $\quad f'(y) = -8y \Rightarrow y=0\ (0,0)$

$x=1\ \ 0 \le y \le 1$ $\quad f(1,y) = 2-4y^2$ $\quad f'(y) = -8y \Rightarrow y=0\ (1,0)$

$f(0,0) = 0$ $\quad \underset{\text{min}}{f(0,1) = -4}$ $\quad \underset{\text{max}}{f(1,0) = 2}$

<u>Q5</u> max $f(x,y) = x^2y + 2x + y$ $\quad \nabla f = \lambda \nabla g$

subject to $g(x,y) = xy - 4 = 0$ $\quad g = 0$

$\nabla f = \langle 2xy + 2, x^2 + 1 \rangle$

$\nabla g = \langle y, x \rangle$

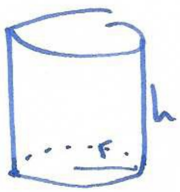
$$\left.\begin{aligned} 2xy + 2 &= \lambda y \\ x^2 + 1 &= \lambda x \\ xy &= 4 \end{aligned}\right] \rightarrow 10 = \lambda y \qquad \lambda = \frac{10}{y} = \frac{10x}{4} = \frac{5x}{2}$$

$x^2+1 = \frac{5x^2}{2}$ $\quad 3x^2 = 2 \quad x = \pm\sqrt{2/3} \quad y = \pm 4/\sqrt{2/3}$

$f\left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{4\sqrt{3}}{\sqrt{2}}\right) = \frac{2}{3\,9}\cdot\frac{4\sqrt{3}}{\sqrt{2}} + \frac{2\sqrt{2}}{\sqrt{3}} + \frac{4\sqrt{3}}{\sqrt{2}}.$ max

$f\left(-\frac{\sqrt{2}}{\sqrt{3}}, -\frac{4\sqrt{3}}{\sqrt{2}}\right) = \frac{2}{9}\frac{4\sqrt{3}}{\sqrt{2}} - \frac{2\sqrt{2}}{\sqrt{3}} - \frac{4\sqrt{3}}{\sqrt{2}}.$ min.

Q6



$V = \pi r^2 h$

$A = 2\pi r h + 2\pi r^2$

min $A = 2\pi r h + 2\pi r^2$

subject to $V = \pi r^2 h = V_0$.

$\nabla A = \lambda \nabla V$

$V = V_0$

$\nabla A = \langle 2\pi r, 2\pi h + 4\pi r \rangle$

$\nabla V = \langle \pi r^2, 2\pi r h \rangle$

$2\pi r = \lambda \pi r^2$

$2\pi h + 4\pi r = \lambda 2\pi r h$

$\pi r^2 h = V_0$.

$\left.\begin{array}{l} 2 = \lambda r \\ \pi h + 2\pi r = \lambda \pi r h \end{array}\right\}$

$\pi r^2 h = V_0$

$\cancel{\pi}h + 2\cancel{\pi}r = 2\cancel{\pi}h$

$h + 2r = 2h$

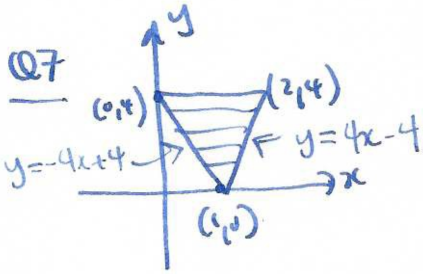
$2r = h$.

$\pi r^2 \cdot 2r = V_0$.

$r = \sqrt[3]{\frac{V_0}{2\pi}}$.

$h = 2\sqrt[3]{V_0/2\pi}$.

Q7



$$\int_0^4 \int_{\cancel{-4x+4}\ \frac{y}{-4}+1}^{\cancel{4x-4}\ \frac{y}{4}+1} xy \, dx\, dy$$

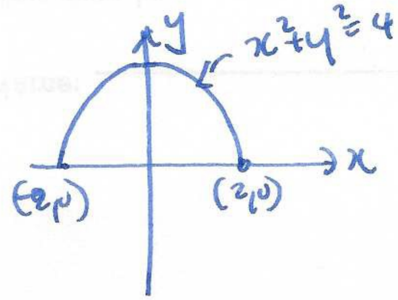
$$\left[\frac{1}{2}x^2\right]_{\frac{y}{-4}+1}^{\frac{y}{4}+1} = \frac{1}{2}\left(\frac{y}{4}+1\right)^2 - \frac{1}{2}\left(1-\frac{y}{4}\right)^2$$

$$= \frac{1}{2}\left[\cancel{\frac{y^2}{16}} + \frac{y}{2} + \cancel{1} - \cancel{1} + \frac{y}{2} - \cancel{\frac{y^2}{16}}\right] = \frac{1}{2}y$$

$$\int_0^4 \frac{1}{2}y\,dy = \left[\frac{1}{4}y^2\right]_0^4 = 4.$$

Q8

$$\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \frac{y}{(1+x^2+y^2)^2}\,dx\,dy$$

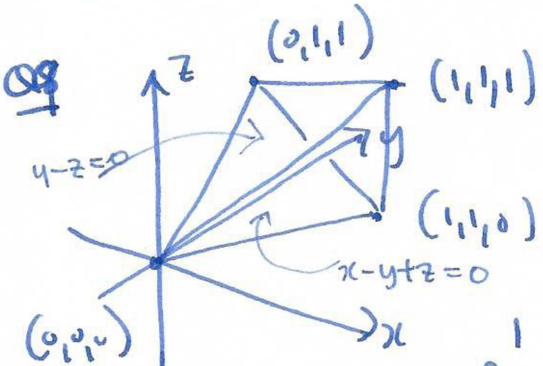


$$\int_{-2}^{2} \underbrace{\int_{0}^{\sqrt{4-x^2}} \frac{y}{(1+x^2+y^2)^2}\, dy}\, dx$$

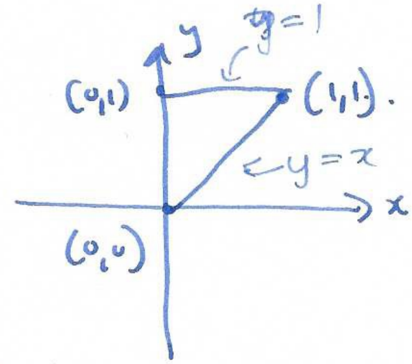
$$\left[-(1+x^2+y^2)^{-1}\frac{1}{2}\right]_0^{\sqrt{4-x^2}} = \frac{-1/2}{1+x^2+4-x^2} + \frac{1/2}{1+x^2} = \frac{1/2}{1+x^2} - \frac{1}{10}.$$

$$\int_{-2}^{2} \frac{1/2}{1+x^2} - \frac{1}{10}\, dx = \left[\frac{1}{2}\tan^{-1}(x) - \frac{x}{10}\right]_{-2}^{2} = \tan^{-1}(2) - \frac{2}{5}$$

Q9

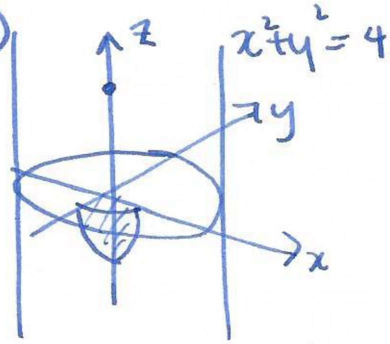


projection into xy-plane

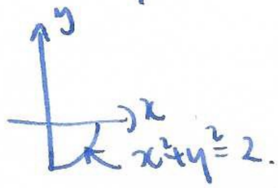


$$\int_0^1 \int_x^1 \int_{y-x}^{y} dz\, dy\, dx$$

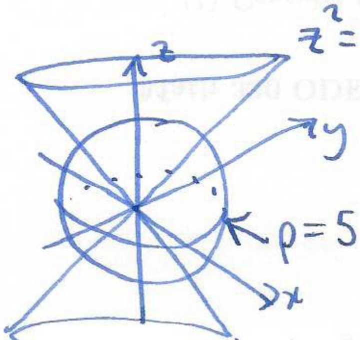
Q10 a)



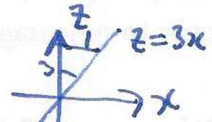
$$\int_0^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{0} \int_{-\sqrt{4-2x^2-2y^2}}^{0} dz\, dy\, dx$$



b)



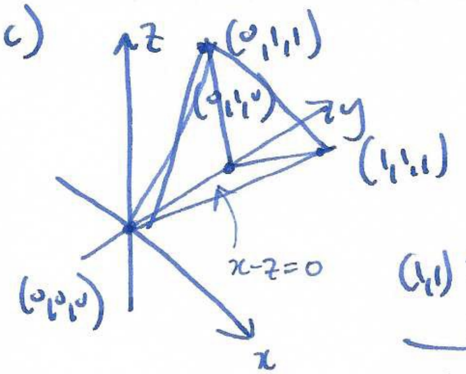
$z^2 = 3x^2+3y^2$



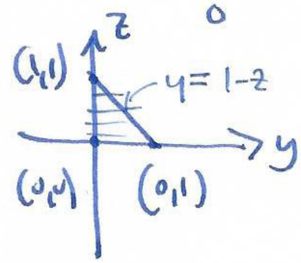
$\theta = \tan^{-1}(1/3)$

$$\int_{\pi-\tan^{-1}(1/3)}^{\pi} \int_{\pi}^{2\pi} \int_0^5 \cancel{4}\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

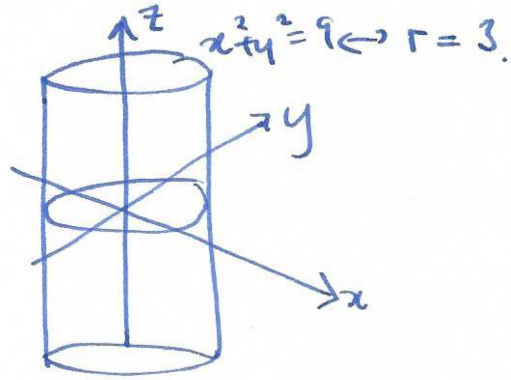
c)



$$\int_0^1 \int_0^{1-z} \int_0^{z} dx\,dy\,dz$$

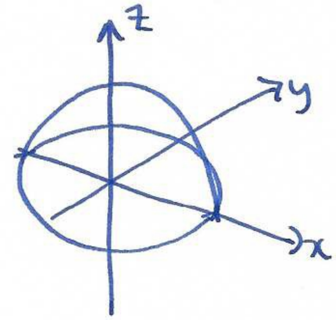


Q11



$$\int_0^{2\pi} \int_0^{3} \int_{-r(\cos\theta+\sin\theta)^2}^{2r(\cos\theta+\sin\theta)^2} r\,dz\,dr\,d\theta$$

Q12



$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^{3} e^{-\rho^3} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^3 e^{-\rho^3} \rho^2\, d\rho \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\, d\phi$$

$$= \left[-\tfrac{1}{3} e^{-\rho^3}\right]_0^3 \cdot 2\pi \cdot \left[-\cos\phi\right]_0^{\pi/2} = \tfrac{1}{3}\left(1-e^{-27}\right) 2\pi \cdot 1 = \tfrac{2\pi}{3}\left(1-e^{-27}\right)$$

Q13 $f(x,y,z) = e^{y} + xz$

$\underline{c}(t) = (-1,2,-2)(1-t) + (3,5,4)t = (-1,2,-2) + t(4,3,6)$

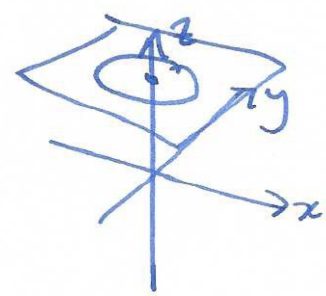
$\underline{c}'(t) = (1,-2,2) + (3,5,4) = (4,3,6)$

$\int_C f\,ds = \int_0^1 \left(e^{2+3t} + (-1+4t)(-2+6t)\right) \|\underline{c}'(t)\|\, dt$

$$= \int_0^1 \left(e^{2+3t} + 2 - 14t + 24t^2\right)\sqrt{16+9+36}\; dt$$

$$= \sqrt{61}\left[\tfrac{1}{3}e^{2+3t} + 2t - 7t^2 + 8t^3\right]_0^1 = \sqrt{61}\left(\tfrac{1}{3}e^5 + 2 - 7 + 8 - \tfrac{1}{3}\right) = \sqrt{61}\left(\tfrac{1}{3}e^5 + \tfrac{8}{3}\right)$$

Q14 $\underline{F} = \langle y^2, x, -z\rangle \qquad \underline{\nabla}\times\underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & -z \end{vmatrix}$



$= \langle 0, 0, 1-2y\rangle \neq \underline{0} \Rightarrow$ not conservative

$\underline{c}(t) = \langle 3\cos t, 3\sin t, 1\rangle \quad 0 \le t \le 2\pi$

$\underline{c}'(t) = \langle -3\sin t, 3\cos t, 0\rangle$

$$\int_C \underline{F}\,d\underline{s} = \int_0^{2\pi} \langle 9\sin^2 t, 3\cos t, -1\rangle \cdot \langle -3\sin t, 3\cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} -27\sin^3 t + 9\cos^2 t\; dt = \int_0^{2\pi} -27\sin t\,(1-\cos^2 t) + \frac{9}{2} + \frac{9}{2}\cos 2t\; dt$$

$$= \left[+27\cos t + -\frac{27\cos^3 t}{3} + \frac{9}{2}t + \frac{9}{4}\sin 2t\right]_0^{2\pi} = 9\pi.$$

Q15 $\underline{F} = \langle ze^{xz}, -z\sin(yz), xe^{xz} - y\sin(yz)\rangle$

$\underline{\nabla}\times\underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{xz} & -z\sin(yz) & xe^{xz} - y\sin(yz) \end{vmatrix} = \langle -\sin(yz) - yz\cos(yz) + \sin(yz) + zy\cos(yz),\; -e^{xz}\cdot z + ze^{xz},\; 0\rangle = \underline{0}.$

$\Rightarrow \underline{F}$ is conservative.

$\frac{\partial f}{\partial x} = ze^{xz} \Rightarrow f = e^{xz} + c_1(y,z)$

$\frac{\partial f}{\partial y} = -z\sin(yz) \Rightarrow f = -\cos(yz) + c_2(x,z)$

$\frac{\partial f}{\partial z} = xe^{xz} - y\sin(yz) \Rightarrow f = e^{xz} - \cos(yz) + c_3(x,y)$

$f(x,y,z) = e^{xz} - \cos(yz) + c$

$\int_{\text{closed curve}} \underline{F}\,d\underline{s} = 0.$