Math 233 Calculus 3 Fall 2015 Final b

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator with no symbolic algebra, but no cell phone. You may bring a US letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

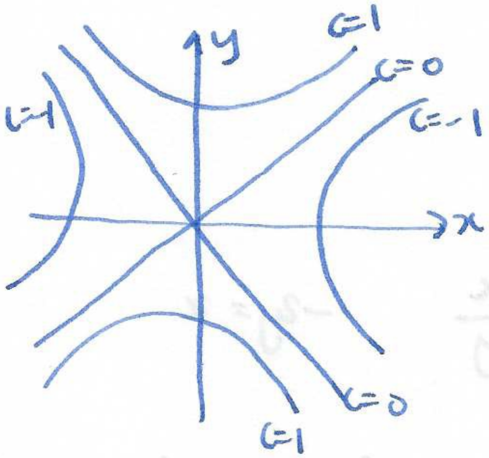
(1) (10 points) Find the volume of the parallelepiped determined by the three vectors $\langle 1, 1, 0\rangle$, $\langle 2, 1, 1\rangle$ and $\langle +1, 1, -2\rangle$.

$$\text{vol} = |a(b \times c)| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ +1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ +1 & -2 \end{vmatrix} \cancel{\pm|-3-3| = \pm 6}$$

$$= -3 - (-4-1) = 2$$

$$\text{vol} = \cancel{6}\ 2$$

(2) Sketch the level sets of the function $f(x, y) = y^2 - x^2$, and calculate the gradient vector at the point $(1, 2)$.



$\nabla f(x,y) = \cancel{\langle 2y, -2x\rangle}\ \langle -2x, 2y\rangle$

$\nabla f(1,2) = \cancel{\langle 4, -2\rangle}\ \langle -2, 4\rangle$

(3) Use Lagrange's method to find the maximum value of the function $f(x, y) = 2x - y$ on the circle $x^2 + y^2 = 9$.

max $f(x,y) = 2x - y$ $\quad \nabla f = \langle 2, -1 \rangle$

subject to $x^2+y^2 = 9$ $\quad \nabla g = \langle 2x, 2y \rangle$

Solve $\nabla f = \lambda \nabla g$

$g = 9$

$$\left.\begin{aligned} 2 &= \lambda 2x \\ -1 &= \lambda 2y \end{aligned}\right\} \quad -2 = \frac{x}{y} \qquad -2y = x$$

$x^2+y^2 = 9$

$\left(\frac{3}{\sqrt5}, \frac{-6}{\sqrt5}\right) \left(\frac{-3}{\sqrt5}, \frac{6}{\sqrt5}\right).$ $\quad \Rightarrow \quad 4y^2+y^2 = 9 \quad y = \pm\frac{3}{\sqrt5}, \; x = \mp\frac{6}{\sqrt5}$

$f\left(\frac{3}{\sqrt5}, \frac{-6}{\sqrt5}\right) = \frac{12}{\sqrt5}$ max

$f\left(-\frac{3}{\sqrt5}, \frac{6}{\sqrt5}\right) = -\frac{12}{\sqrt5}$ min

(4) Find the critical points of $f(x, y) = xe^{2x} - y^2e^{2x}$ and use the second derivative test to classify them.

$\frac{\partial f}{\partial x} = e^{2x} + 2xe^{2x} - 2y^2e^{2x} = 0 \Rightarrow e^{2x}(1+2x) = 0$

$x = -1/2$

$\frac{\partial f}{\partial y} = -2ye^{2x} = 0 \Rightarrow y = 0$

critical point $(-\frac{1}{2}, 0)$

$f_{xx} = 2e^{2x} + 2e^{2x} + 4xe^{2x} - 4y^2e^{2x}$

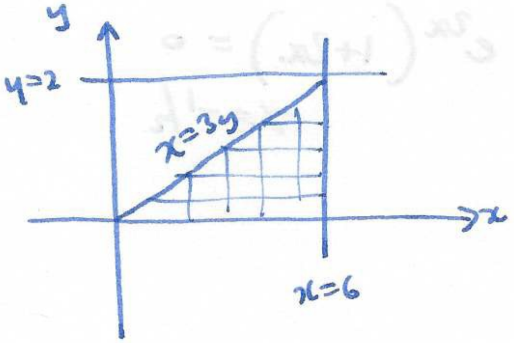
$f_{xy} = -4ye^{2x}$

$f_{yy} = -2e^{2x}$

$D(-\frac{1}{2}, 0) = f_{xx}f_{yy} - (f_{xy})^2 = (4e^{-1} - 2e^{-1})(-2e^{-1}) - 0 = -\frac{4}{e} < 0$

$\Rightarrow$ saddle.

(5) Change the order of integration to evaluate $\int_0^2 \int_{3y}^6 e^{-x^2}\, dx\, dy$.

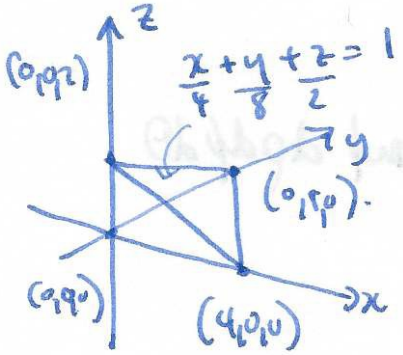


$$\int_0^6 \int_0^{x/3} e^{-x^2}\, dy\, dx$$

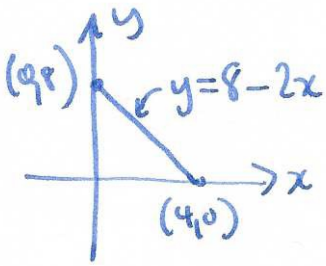
$$\left[y e^{-x^2} \right]_0^{x/3} = \frac{1}{3} x e^{-x^2}$$

$$\int_0^6 \frac{1}{3} x e^{-x^2}\, dx = \left[-\frac{1}{6} e^{-x^2} \right]_0^6 = \frac{1}{6}\left(1 - e^{-36}\right).$$

(6) Use a triple integral to find the volume of the tetrahedron formed with vertices $(0,0,0)$, $(4,0,0)$, $(0,8,0)$ and $(0,0,2)$.



$$\int_0^4 \int_0^{8-2x} \int_0^{2-\frac{x}{2}-\frac{y}{4}} 1 \; dz\,dy\,dx$$



$$\left[z \right]_0^{2-\frac{x}{2}-\frac{y}{4}} = 2-\frac{x}{2}-\frac{y}{4}$$

$$\left[2y - \frac{1}{2}xy - \frac{1}{8}y^2 \right]_0^{8-2x}$$

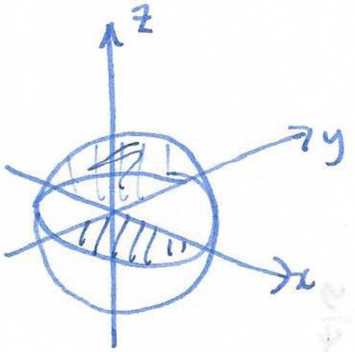
$$= 2(8-2x) - \frac{1}{2}x(8-2x) - \frac{1}{8}\underbrace{(8-2x)^2}_{(64-32x+4x^2)}$$

$$= 16 - 4x - 4x + x^2 - 8 + 4x - \frac{1}{2}x^2$$

$$= 8 - 4x + \frac{1}{2}x^2$$

$$\left[8x - 2x^2 + \frac{1}{6}x^3 \right]_0^4 = 32 - 32 + \frac{1}{6}64 = \frac{32}{3}$$

(7) Write down limits for integrating $f(x, y, z) = xyz$ over the region inside the sphere of radius 3 in the octant $x \geq 0$, $y \leq 0$ and $z \geq 0$. Do not evaluate this integral.



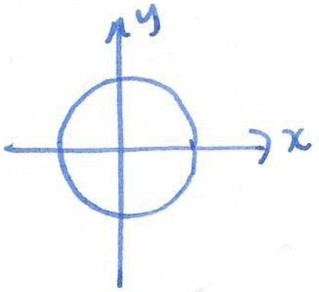
$$\int_{3\pi/2}^{2\pi} \int_0^{\pi/2} \int_0^3 \rho^3 \sin\theta\cos\theta \sin^2\phi \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$x = \rho\cos\theta \overset{\sin}{\cancel{\cos}}\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

(8) Let C be the unit circle with anticlockwise orientation, and let $\mathbf{F} = \langle \tan^{-1}(x^2), xy^2, \rangle$. Use Green's Theorem to evaluate $\int_C \mathbf{F}\, ds$.



$$\int_C \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint y^2 - 0\, dA$$

$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 r^2 \sin^2\theta \;\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} \underbrace{\sin^2\theta}_{= \frac{1}{2} - \frac{1}{2}\cos 2\theta}\, d\theta \int_0^1 r^3\, dr$$

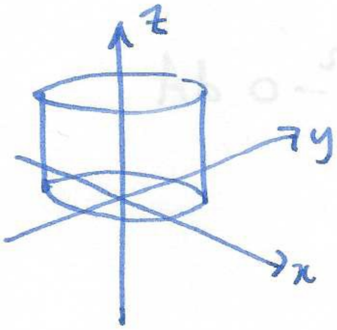
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

$$\left[\tfrac{1}{4} r^4\right]_0^1 = \frac{1}{4}$$

$$\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi.$$

$$\int_C \underline{F}\cdot d\underline{s} = \frac{\pi}{4}$$

(9) Let $\mathbf{F} = \langle 2yz, -2xz, \sin(z^2)\rangle$. Let S be the part of the cylinder $x^2 + y^2 = 4$, with $0 \le z \le 1$, with the outward pointing normal. Use Stokes' Theorem to evaluate $\displaystyle\int_{\partial S} \mathbf{F}\, d\mathbf{s}$.



$$\int_{\partial S} \underline{F}\cdot \underline{ds} = \iint_S \underline{\nabla}\times\underline{F}\cdot \underline{dS}$$

$$\underline{\nabla}\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & -2xz & \sin(z^2) \end{vmatrix} = \langle +2x, 2y, -2z-2z\rangle$$

$$\Phi(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle \qquad 0 \le \theta \le 2\pi \quad 0 \le z \le 1$$

$$\Phi_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle \qquad \underline{n} = \Phi_\theta \times \Phi_z = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

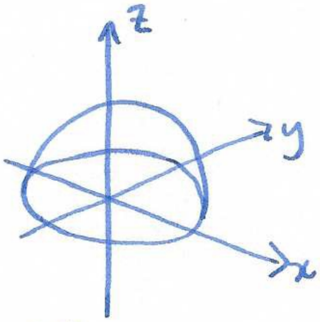
$$\Phi_z = \langle 0, 0, 1\rangle$$

$$= \langle 2\cos\theta, 2\sin\theta, 0\rangle$$

$$\int_0^1 \int_0^{2\pi} \langle +4\cos\theta, 4\sin\theta, -4z\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle \, d\theta\, dz$$

$$\int_0^1 \int_0^{2\pi} 8\cos^2\theta + 8\sin^2\theta \, d\theta\, dz = 8\int_0^1 dz \int_0^{2\pi} d\theta = 16\pi.$$

(10) Let W be the hemisphere $x^2 + y^2 + z^2 \leq 4$ with $z \geq 0$, and let $\mathbf{F} = \langle x, y, z + \ln(x^2+1)\rangle$. Use the divergence theorem to evaluate $\displaystyle\int\int_{\partial W} \mathbf{F}\, d\mathbf{S}$.



$$\iint_{\partial W} \underline{F} . d\underline{S} = \iiint_W \underline{\nabla} . \underline{F} . dV$$

$$\underline{\nabla} . \underline{F} = 1 + 1 + 1 = 3.$$

$$\int_0^{\pi/2} \int_0^{2\pi} \int_0^2 3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = 3 \int_0^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho.$$

$$\left[-\cos\phi\right]_0^{\pi/2} = 1 - 0 \qquad \left[\tfrac{1}{3}\rho^3\right]_0^2 = \frac{8}{3}.$$

$$\iint_{\partial W} \underline{F} . d\underline{S} = 3 . 1 . 2\pi . \frac{8}{3} = 16\pi.$$