Math 233 Calculus 3 Fall 2015 Final a

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator with no symbolic algebra, but no cell phone. You may bring a US letter page of notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

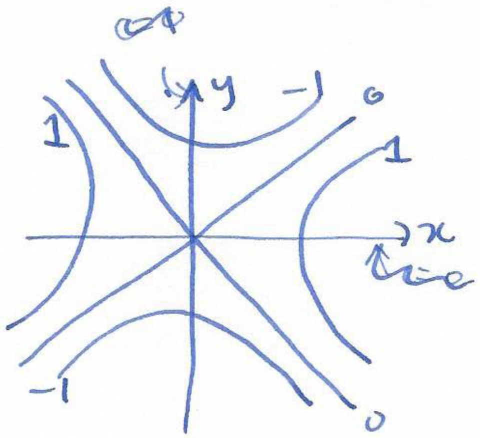
Final	
Overall	

(1) (10 points) Find the volume of the parallelepiped determined by the three vectors $\langle 0,1,1\rangle$, $\langle 1,2,1\rangle$ and $\langle 1,-1,+2\rangle$.

$$\text{vol} = |\underline{a}\cdot(\underline{b}\times\underline{c})| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & +2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & +2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= |-1-3| = 4$$

(2) Sketch the level sets of the function $f(x, y) = x^2 - y^2$, and calculate the gradient vector at the point $(2, 1)$.



$x^2 - y^2 = c$

$\nabla f = \langle 2x, -2y \rangle$

$\nabla f(2,1) = \langle 4, -2 \rangle$

(3) Use Lagrange's method to find the maximum value of the function $f(x, y) = x - 2y$ on the circle $x^2 + y^2 = 4$.

max $f(x,y) = x - 2y$ $\qquad \nabla f = \langle 1, -2 \rangle$

subject to $g(x,y) = x^2 + y^2 = 4$ $\qquad \nabla g = \langle 2x, 2y \rangle$

solve $\nabla f = \lambda \nabla g$

$$\left.\begin{aligned} 1 &= \lambda 2x \\ -2 &= \lambda 2y \end{aligned}\right\} \qquad -\frac{1}{2} = \frac{x}{y} \qquad y = -2x$$

$$x^2 + y^2 = 4$$

$\Rightarrow$ $\quad x^2 + 4x^2 = 4 \qquad x = \pm \frac{2}{\sqrt{5}} \qquad \left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) \quad \left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$

$f\left(\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}}$ max

$f\left(-\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = -\frac{2}{\sqrt{5}} - \frac{8}{\sqrt{5}} = \frac{-10}{\sqrt{5}}$ min

(4) Find the critical points of $f(x,y) = ye^{3y} - x^2 e^{3y}$ and use the second derivative test to classify them.

$\frac{\partial f}{\partial x} = -2xe^{3y} = 0 \Rightarrow x = 0$

$\frac{\partial f}{\partial y} = e^{3y} + 3ye^{3y} - 3x^2 e^{3y}$

$e^{3y} + 3ye^{3y} = 0 \Rightarrow y = -1/3.$

$e^{3y}(1+3y) = 0$

critical point $(0, -\frac{1}{3})$

$f_{xx} = -2e^{3y}$

$f_{xy} = -6xe^{3y}$

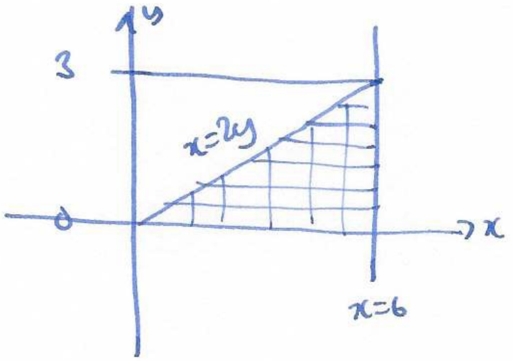
$f_{yy} = 3e^{3y} + 3e^{3y} + 9ye^{3y} - 9x^2 e^{3y}$

$D(0, -\frac{1}{3}) = f_{xx} f_{yy} - (f_{xy})^2 = \cancel{(-2e^{-1}) \cdot (0) - (6e^{-1} - 3e^{-1})^2} < 0$

$\Rightarrow$ saddle

$(-2e^{-1})(3e^{-1}) - 0 < 0$

(5) Change the order of integration to evaluate $\int_0^3 \int_{2y}^6 e^{-x^2}\, dx\, dy$.

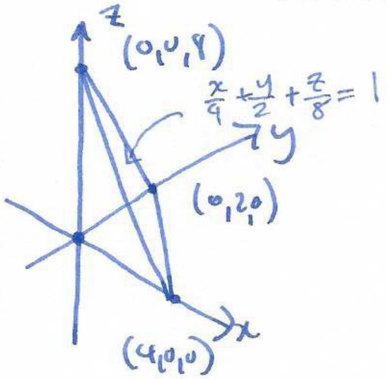


$$\int_0^6 \int_0^{x/2} e^{-x^2}\, dy\, dx$$

$$\left[y e^{-x^2} \right]_0^{x/2} = \frac{x}{2} e^{-x^2}$$

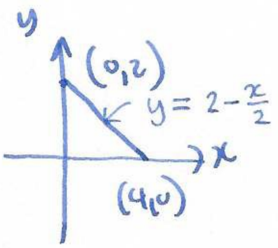
$$\int_0^6 \frac{1}{2} x e^{-x^2}\, dx = \left[-\frac{1}{4} e^{-x^2} \right]_0^6 = \frac{1}{4}\left(1 - e^{-36}\right)$$

(6) Use a triple integral to find the volume of the tetrahedron formed with vertices $(0,0,0)$, $(4,0,0)$, $(0,2,0)$ and $(0,0,8)$.



$$\int_0^4 \int_0^{2-\frac{x}{2}} \int_0^{8-2x-4y} 1 \, dz\, dy\, dx$$

$$\left[z\right]_0^{8-2x-4y} = 8-2x-4y$$



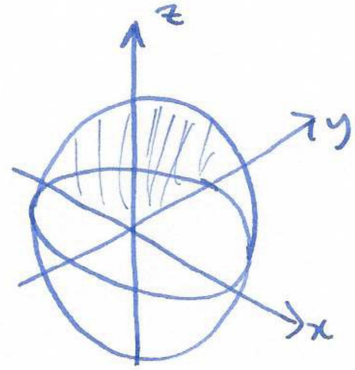
$$\int_0^{2-\frac{x}{2}} 8-2x-4y \, dy = \left[8y - 2xy - 2y^2\right]_0^{2-\frac{x}{2}}$$

$$= 8\left(2-\frac{x}{2}\right) - 2x\left(2-\frac{x}{2}\right) - 2\left(2-\frac{x}{2}\right)^2$$

$$= 16 - 4x - 4x + x^2 - 8 + 4x - \frac{x^2}{2} = 8 - 4x + \frac{x^2}{2}$$

$$\int_0^4 8 - 4x + \frac{x^2}{2} \, dx = \left[8x - 2x^2 + \frac{x^3}{6}\right]_0^4 = 32 - 32 + \frac{64}{6} = \frac{64}{6} = \frac{32}{3}.$$

(7) Write down limits for integrating $f(x, y, z) = xyz$ over the region inside the sphere of radius 3 in the octant $x \leq 0$, $y \geq 0$ and $z \geq 0$. Do not evaluate this integral.



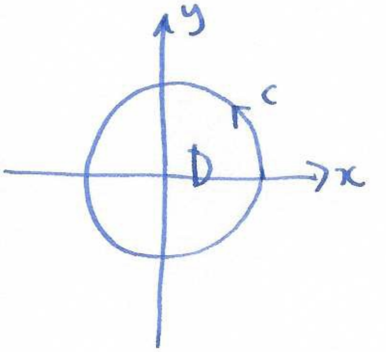
$$\int_0^3 \int_{\pi/2}^{\pi} \int_0^{\pi/2} \rho^3 \sin\theta \cos\theta \sin^2\phi \cos\phi \; \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$x = \rho \cos\theta \sin\phi$

$y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi$

(8) Let C be the unit circle with anticlockwise orientation, and let $\mathbf{F} = \langle x^2 y, \tan^{-1}(y^2)\rangle$. Use Green's Theorem to evaluate $\int_C \mathbf{F}\, d\mathbf{s}$.



$$\int_C \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\, dA$$

$$= \iint_D 0 - x^2\, dA$$

use polars
$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^1 \int_0^{2\pi} -r^2\cos^2\theta\, r\, d\theta\, dr = -\int_0^1 r^3\, dr \int_0^{2\pi} \cos^2\theta\, d\theta$$

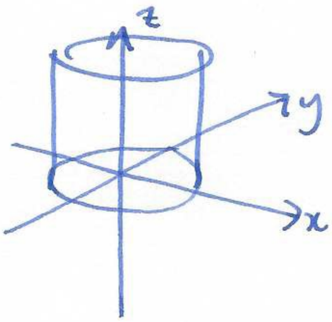
$\frac{1}{2} + \frac{1}{2}\cos 2\theta$ //

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$

$$\left[\frac{1}{4}r^4\right]_0^1 = \frac{1}{4} \qquad \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi$$

$$\int_C \underline{F}\cdot d\underline{s} = -\frac{1}{4}\pi.$$

(9) Let $\mathbf{F} = \langle 3yz, -3xz, \sin(z^2)\rangle$. Let S be the part of the cylinder $x^2 + y^2 = 1$, with $0 \leq z \leq 2$, with the outward pointing normal. Use Stokes' Theorem to evaluate $\displaystyle\int_{\partial S} \mathbf{F}\; d\mathbf{s}$.



$$\int_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S \underline{\nabla\times F}\cdot d\underline{S}$$

$$\underline{\nabla\times F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3yz & -3xz & \sin(z^2) \end{vmatrix} = \langle +3x, 3y, -3z-3z\rangle$$

parameterization: $\Phi(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle$ $\quad 0 \leq \theta \leq 2\pi \quad 0 \leq z \leq 2$

$\Phi_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$

$\Phi_z = \langle 0, 0, 1\rangle$

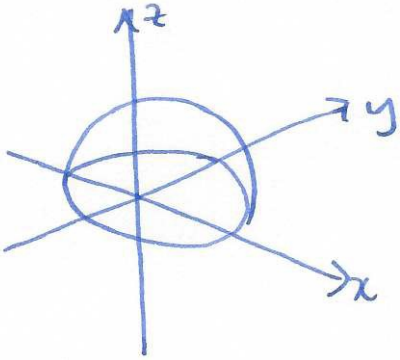
$$\underline{n} = \Phi_\theta \times \Phi_z = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0\rangle$$

$$\int_0^2 \int_0^{2\pi} \langle 3\cos\theta, 3\sin\theta, -6z\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle \, d\theta\, dz$$

$$= \int_0^2 \int_0^{2\pi} \underbrace{3\cos^2\theta + 3\sin^2\theta}_{3} \, d\theta\, dz = \Big[3\theta\Big]_0^{2\pi} = 6\pi$$

$$\int_0^2 6\pi \, dz = \Big[6\pi z\Big]_0^2 = 12\pi.$$

(10) Let W be the hemisphere $x^2+y^2+z^2 \leq 4$ with $z \geq 0$, and let $\mathbf{F} = \langle x, y + \ln(z^2+1), z\rangle$. Use the divergence theorem to evaluate $\displaystyle\iint_{\partial W} \mathbf{F}\, d\mathbf{S}$.



$$\iint_{\partial W} \underline{F} \cdot d\underline{S} = \iiint_W \underline{\nabla} \cdot \underline{F} \, dV$$

$$\underline{\nabla} \cdot \underline{F} = 1 + 1 + 1 = 3.$$

$$\int_0^{\pi/2}\int_0^{2\pi}\int_0^2 3\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = 3 \int_0^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} \cancel{\sin}\, d\theta \int_0^2 \rho^2\, d\rho$$

$$\Big[-\cos\phi\Big]_0^{\pi/2} = 1 \qquad \Big[\tfrac{1}{3}\rho^3\Big]_0^2 = \frac{8}{3}. \qquad \int_0^{2\pi} d\theta = 2\pi$$

$$\iint_{\partial W} \underline{F} \cdot d\underline{S} = 3 \cdot 1 \cdot \frac{8}{3} \cdot 2\pi = 16\pi.$$