induction step $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$

$H^*(\mathbb{CP}^{n-1}) \leftarrow H^*(\mathbb{CP}^n)$ so $H^*(\mathbb{CP}^{n}) \cong \begin{array}{cccccccc} 0 & 1 & 2 & \cdots & 2n-3 & 2n-2 & 2n-1 & 2n \\ \mathbb{Z} & 0 & \mathbb{Z} & \cdots & 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ 1 & & \alpha & \cdots & & \alpha^{n-1} & & ? \end{array}$

then by non-singularity, given $\alpha \in H^2(\mathbb{CP}^n)$ there is $m\alpha^{n-1} \in H^{2n-2}(\mathbb{CP}^n)$ s.t. $\alpha \cup m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{CP}^n)$ $\Rightarrow m = \pm 1$ $\square$.

$H^*(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$: same argument.

<u>Other forms of duality</u>

<u>recall</u> M^n n-manifold with boundary if each $x \in M^n$ has an open nbd homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$
points w/ nbds homeomorphic to $\mathbb{R}^n_+$ form the boundary of M: ∂M.

<u>Def^n</u> A collar neighborhood of ∂M is an open neighborhood homeo to $\partial M \times [0,1)$

<u>Prop^n</u> If M is a compact manifold w/ boundary, then ∂M has a collar nbd $\square$

<u>Relative fundamental class</u> if M is orientable, then $[M]$ a generator of $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ is a relative fundamental class. If M is a simplicial complex $[M]$ consists of one copy of each simplex with an appropriate choice of sign.

<u>Th^m</u> (Poincaré–Lefschetz duality) M compact orientable w/ boundary, then

$D_M : H^k(M, \partial M; \mathbb{Z}) \to H_{n-k}(M; \mathbb{Z})$ is an isomorphism and
$\phi \longmapsto [M] \frown \phi$.

$D_M : H^k(M; \mathbb{Z}) \to H_{n-k}(M, \partial M; \mathbb{Z})$ also iso. Fancy version $\partial M = A \cup B$ compact w/ $A \cap B = \partial A = \partial B$

then $D_M : H^k(M, A; \mathbb{Z}) \longrightarrow H_{n-k}(M, B; \mathbb{Z})$ is an isomorphism [works w/ field coeffs]
$\phi \longmapsto [M] \frown \phi$

Alexander duality

Thm K compact, locally contractible, non-empty proper subspace of S^n, then $\tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}^{n-i-1}(K; \mathbb{Z})$ for all i.

Special case $K = S^1$ $\quad H^*(S^1) = \begin{array}{ccc} 0 & 1 & \geq 2 \\ \mathbb{Z} & \mathbb{Z} & 0 \end{array}$.

so $\tilde{H}_1(S^3 \setminus K) \cong \tilde{H}^1(S^1) \cong \mathbb{Z}$

$\tilde{H}_2(S^3 \setminus K) \cong \tilde{H}^0(S^1) \cong 0$.

Proof

$$H_i(S^n \setminus K; \mathbb{Z}) \cong H_c^{n-i}(S^n \setminus K) \quad \text{(Poincaré duality)}$$

$$= \varinjlim_{U \text{ open nbd of } K} H^{n-i}(S^n \setminus K, U \setminus K) \quad \text{(defn of cohomology w/ compact supports)}$$

$$\cong \varinjlim H^{n-i}(S^n, U) \quad \text{(excision)}$$

$$\cong \varinjlim \tilde{H}^{n-i-1}(U) \quad \text{(long exact sequence of a pair)}$$

$$\cong \tilde{H}^{n-i-1}(K) \quad \text{(assume collar nbd)}.$$