

Lemma $M = U \cup V$ open then

$$\begin{array}{ccccccc}
 \dots \rightarrow H_c^k(U \cap V) & \rightarrow & H_c^k(U) \oplus H_c^k(V) & \rightarrow & H_c^k(M) & \rightarrow & H_c^{k+1}(U \cap V) \rightarrow \dots \\
 \downarrow D_{U \cap V} & & \downarrow D_U \oplus D_V & & \downarrow D_M & & \downarrow D_{U \cap V} \\
 \dots \rightarrow H_{n-k}(U \cap V) & \rightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \rightarrow & H_{n-k}(M) & \rightarrow & H_{n-k-1}(U \cap V) \rightarrow \dots
 \end{array}$$

commutes up to sign.

Proof (of Poincaré duality using Lemma)

a) if $M = U \cup V$ open, and $D_U, D_V, D_{U \cap V}$ isomorphisms $\Rightarrow D_M$ iso by five lemma.

b) if $M = \bigcup U_i$ and each D_{U_i} is an iso, take limit.

$$H_c^k(U_i) = \lim_{\substack{\leftarrow \\ K \subset U_i \\ \text{compact}}} H_c^k(M/K), \text{ get } H_c^k(U_i) \rightarrow H_c^k(U_{i+1}) \text{ so we can take } \lim_{\leftarrow U_i} H_c^k(U_i) \cong H_c^k(M).$$

and for homology $H_{n-k}(M) \cong \lim_{\leftarrow} H_{n-k}(U_i)$

so D_M is limit of isomorphisms D_{U_i} , so is an isomorphism.

1) $M = \mathbb{R}^n = \text{int}(\Delta^n)$ $D_M : H_c^k(\Delta^n, \partial \Delta^n) \rightarrow H_{n-k}(\Delta^n)$
 $\alpha \mapsto [\Delta^n] \cap \alpha$

only non-zero for $k=n$, when this is an iso

as generator of $H^n(\Delta^n, \partial \Delta^n)$ is $\phi : \Delta^n \rightarrow 1$, so $[\Delta^n] \cap \phi = [v_{n+1}]$ generates $H_0(\Delta^n)$

2) $M \subset \mathbb{R}^n$: write M as a countable union of (convex) balls B_i

and set $V_i = \bigcup_{j \leq i} B_j$ apply a) to $U_i = V_i$ $U_i \cap V_i$ is convex, induction of number of convex sets, now $M = \bigcup V_i$ apply b).

3) M countably union of open sets, apply b).

Corollary A closed manifold of odd dimension has Euler characteristic zero. (88)

Proof: M^n orientable: $\text{rank } H_i(M; \mathbb{Z}) = \text{rank } H^{n-i}(M; \mathbb{Z})$ n odd: cancel in pairs.

M^n not-orientable: double cover, χ multiplicative under covers. \square .

Lemma proof: Mayer-Vietoris; \exists ϕ, ψ ; \exists $\text{homotopic subdivision}$ \square .

Connection with cup product

(*)
$$\psi(\alpha \smile \phi) = (\phi \cup \psi)(\alpha)$$

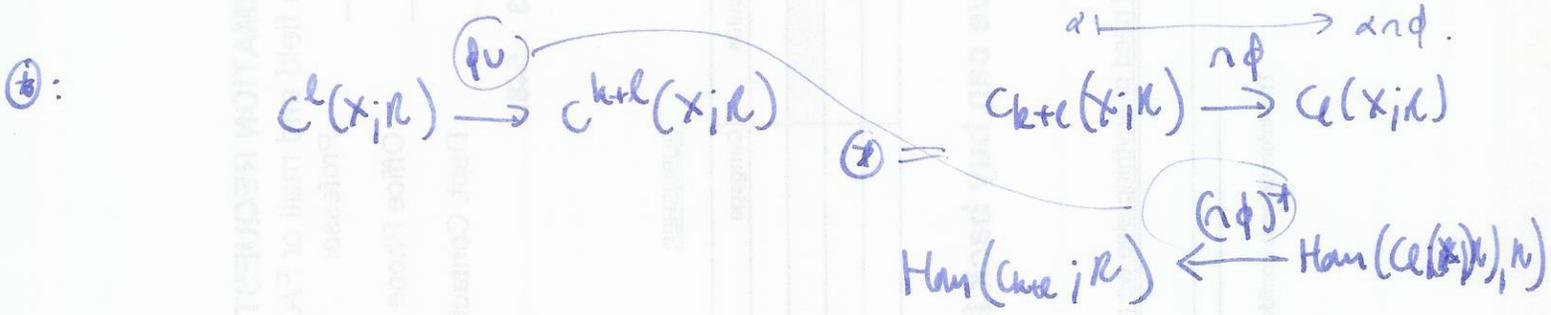
$$\begin{matrix} \smile & \cup \\ \downarrow & \downarrow \\ \ell & \ell \end{matrix} \quad \begin{matrix} \downarrow & \cup \\ \downarrow & \downarrow \\ k+\ell & k+\ell \end{matrix}$$

$$\begin{matrix} \alpha \in C^{k+\ell}(X; \mathbb{R}) \\ \phi \in C^k(X; \mathbb{R}) \\ \psi \in C^\ell(X; \mathbb{R}) \end{matrix}$$

check: $\sigma: \Delta^{k+\ell} \rightarrow X$

$$\psi(\sigma \smile \phi) = \psi(\phi(\sigma|_{[v_0, \dots, v_k]}) \sigma|_{[v_{k+1}, \dots, v_{k+\ell}]})$$

$$= \psi(\phi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_{k+1}, \dots, v_{k+\ell}]}) = \text{det}(\sigma) \cdot \square$$



so
$$\begin{matrix} H^l(X; \mathbb{R}) & \xrightarrow{h} & \text{Hom}(H^l(X; \mathbb{R}), \mathbb{R}) \\ \downarrow \phi_U & & \downarrow (\psi \circ \phi)^* \\ H^{k+l}(X; \mathbb{R}) & \xrightarrow{h} & \text{Hom}(H^{k+l}(X; \mathbb{R}), \mathbb{R}) \end{matrix}$$
 commutes.

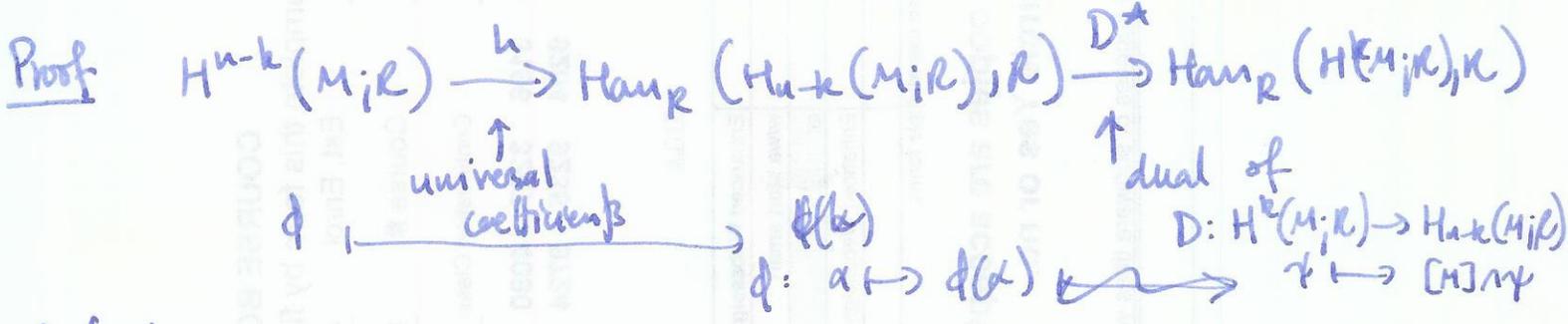
in fact $\phi_U = (\psi \circ \phi)^*$ as homomorphisms / chains when $R = \text{field}$ or $R = \mathbb{Z}$ as fraction.

M closed orientable $H^k(M;R) \times H^{n-k}(M;R) \rightarrow R$

$$\phi, \psi \mapsto (\phi \cup \psi)[M]$$

bilinear, non-singular: $A \times B \rightarrow R$ if $A \rightarrow \text{Hom}(B, R)$ and $B \rightarrow \text{Hom}(A, R)$ are isomorphisms.

Propⁿ The cup product pairing is non-singular for closed R -orientable manifolds when R is a field, or $R = \mathbb{Z}$ and you factor out torsion in $H^*(M; \mathbb{Z})$.



R field \mathbb{Z} H^k H_n $\Rightarrow h$ is an isomorphism.

$$1 \mapsto \psi \mapsto \phi(\alpha) \cap \psi$$

D iso $\Rightarrow D^*$ iso: gives non-singularity in one factor, other factor follows from commutativity \square .

Corollary If M is a closed connected orientable n -manifold, then for any $\alpha \in H^k(M; \mathbb{Z})$ of infinite order, primitive, there is a $\beta \in H^{n-k}(M; \mathbb{Z})$ s.t.

$\alpha \cup \beta$ is a generator of $H^n(M; \mathbb{Z}) \cong \mathbb{Z}$.

with field coeffs work for any $\alpha \in H^k(M; R)$.

Example Projective spaces

$$H^k(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z} \quad 0 \leq 2k \leq 2n$$

0 otherwise.

and $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces isomorphism on H^* for $k \leq 2n-2$.

$$\textcircled{1} \left[\cong \mathbb{Z}[x] / \langle x^{n+1} \rangle \quad (x) = 2 \text{ as only (not cup product)}. \right]$$

proof of $\textcircled{1}$: induction, base case $H^*(\mathbb{C}P^2)$

0	1	2	3	4
\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}
		\downarrow		
		\mathbb{Z}		

if x generates $H^2(\mathbb{C}P^2)$, then non-singularity $\Rightarrow x^2$ generates $H^4(\mathbb{C}P^2)$