Lemma $M = U \cup V$ open then

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

commutes up to sign.

Proof (of Poincaré duality using Lemma)

a) if $M = U \cup V$ open, and $D_U, D_V, D_{U\cap V}$ isomorphisms $\Rightarrow D_M$ iso by five lemma.

b) if $M = \bigcup U_i$ and each D_{U_i} is an iso, take limits.

$H^k_c(U_i) = \varinjlim_{\substack{K \subset U_i \\ \text{compact}}} H^k(M|K)$, get $H^k_c(U_i) \to H^k_c(U_{i+1})$ so we can take $\varinjlim_{U_i} H^k_c(U_i) \cong H^k_c(M)$.

and for homology $H_{n-k}(M) \cong \varinjlim H_{n-k}(U_i)$

so D_M is limit of isomorphisms D_{U_i}, so is an isomorphism.

1) $M = \mathbb{R}^n = \operatorname{int}(\Delta^n)$ $\quad D_M : H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$

$$\alpha \longmapsto [\Delta^n] \cap \alpha$$

only non-zero for $k = n$, when this is an iso

as generator of $H^n(\Delta^n, \partial\Delta^n)$ is $\phi : \Delta^n \to 1$, so $[\Delta^n] \cap \phi = [v_{n+1}]$ generates $H_0(\Delta^n)$

2) $M \subset \mathbb{R}^n$ open : write M as a countable union of (convex) balls. B_i

and set $V_i = \bigcup_{i \le j} B_j$ apply a) to $U_{i+1} V_{i+1}$ $U_{i+1} V_i$ is convex, induction of number of convex sets, now $M = \bigcup V_i$ apply b).

3) M countably union of open sets, apply b).

Corollary A closed manifold of odd dimension has Euler characteristic zero.

Proof: M^n orientable: rank $H_i(M;\mathbb{Z})$ = rank $H^{n-i}(M;\mathbb{Z})$ n odd: cancel in pairs.

M^n not-orientable: double cover, χ multiplicative under covers. $\square$.

Lemma proof: Mayer–Vietoris; by excision; limits; barycentric subdivision $\square$.

Connection with cup product

$$(*)\quad \underset{l}{\psi}(\alpha \cap \underset{l}{\phi}) = (\phi \cup \psi)(\alpha) \qquad \alpha \in C_{k+l}(X;R)$$

$k+l \qquad k+l.$

$$\phi \in C^k(X;R)$$
$$\psi \in C^l(X;R)$$

check: $\sigma : \Delta^{k+l} \to X$

$$\psi(\sigma \cap \phi) = \psi\big(\phi(\sigma|[v_0,\dots,v_k])\,\sigma|[v_k,\dots,v_{k+l}]\big)$$

$$= \cancel{\psi}\ \phi(\sigma|[v_0,\dots,v_k])\,\psi(\sigma|[v_k,\dots,v_{k+l}]) = \phi\cup\psi(\sigma).\ \square$$

$(*)$:

$$C^l(X;R) \xrightarrow{\phi\cup} C^{k+l}(X;R) \qquad\qquad C_{k+l}(X;R) \xrightarrow{\cap\phi} C_l(X;R)$$

$$\alpha \longmapsto \alpha\cap\phi.$$

$(*)$ =

$$\mathrm{Hom}(C_{k+l};R) \xleftarrow{(\cap\phi)^*} \mathrm{Hom}(C_l(X),R)$$

so

$$\begin{tikzcd}
H^l(X;R) \arrow[r, "h"] \arrow[d, "\phi\cup"] & \mathrm{Hom}(H_l(X;R),R) \arrow[d, "(\cap\phi)^*"] \\
H^{k+l}(X;R) \arrow[r, "h"] & \mathrm{Hom}(H_{k+l}(X;R);R)
\end{tikzcd}$$

commutes.

in fact $\phi\cup = (\cap\phi)^*$ on homology/cohomology when R = field

$R = \mathbb{Z}$ no torsion.

M closed orientable $\quad H^k(M;R)\times H^{n-k}(M;R)\longrightarrow R$

$$\phi\ ,\ \psi \longmapsto (\phi\cup\psi)[M]$$

bilinear, non-singular: $A\times B\to R$ if $A\to \mathrm{Hom}(B,R)$, $B\to \mathrm{Hom}(A,R)$ isomorphisms.

Prop^n. The cup product pairing is non-singular for closed R-orientable manifolds when R is a field, or $R=\mathbb{Z}$ and you factor out torsion in $H^*(M;\mathbb{Z})$.

Proof $\quad H^{n-k}(M;R)\xrightarrow{h}\mathrm{Hom}_R(H_{n-k}(M;R),R)\xrightarrow{D^*}\mathrm{Hom}_R(H^k(M;R),R)$

h: universal coefficients; D^*: dual of $D: H^k(M;R)\to H_{n-k}(M;R)$, $\psi\mapsto [M]\cap\psi$

$\phi \longmapsto \phi(\alpha)$, $\quad \phi:\alpha\mapsto\phi(\alpha)$

$\longmapsto \psi\mapsto \phi([M]\cap\psi) = (\phi\cup\psi)[M]$

R field, $\mathbb{Z}$ H^* mod t, H_* mod t $\Rightarrow h$ is an isomorphism.

D iso $\Rightarrow D^*$ iso : gives non-singularity in one factor, other factor follows from commutativity $\Box$.

Corollary If M is a closed connected orientable n-manifold, then for any $\alpha\in H^k(M;\mathbb{Z})$ of infinite order, primitive, there is a $\beta\in H^{n-k}(M;\mathbb{Z})$ s.t. $\alpha\cup\beta$ is a generator of $H^n(M;\mathbb{Z})\cong\mathbb{Z}$.

With field coeffs works for any $\alpha\in H^k(M;R)$.

Example Projective spaces

$H^*(\mathbb{C}P^n;\mathbb{Z})\cong \mathbb{Z}$ $\ 0\le 2k\le 2n$, $\ 0$ otherwise, and $\mathbb{C}P^{n-1}\hookrightarrow\mathbb{C}P^n$ induces isomorphisms on H^* for $k\le 2n-2$.

(*) $\left[\ \cong \mathbb{Z}[\alpha]/\alpha^{n+1}\quad |\alpha|=2 \ \text{ as a ring (w.r.t. cup product)}\right]$.

proof (of (*)) : induction, base case $H^*(\mathbb{C}P^2)$

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4\\ \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z}\\ 1 & & \alpha & & \end{array}$$

if α generates $H^2(\mathbb{C}P^2)$, then non-singularity $\Rightarrow$ $\alpha\cup\alpha$ generates $H^4(\mathbb{C}P^2)$