Example $X=\mathbb{R}$

~~tensor~~ constant functions

$$\begin{array}{ccccccc} 0 & \leftarrow & \Delta^1(\mathbb{R};\mathbb{Z}) & \xleftarrow{\delta} & \Delta^0(\mathbb{R};\mathbb{Z}) & \leftarrow & 0 \\ & & \cup & & \cup & & \\ 0 & \leftarrow & \Delta^1_c(\mathbb{R};\mathbb{Z}) & \xleftarrow{\delta} & \Delta^0_c(\mathbb{R};\mathbb{Z}) & \leftarrow & 0 \end{array}$$

~~$H^c(\mathbb{R};\mathbb{Z})$~~

$H^0(\mathbb{R};\mathbb{Z}) = \ker\delta/0$ $\quad \delta\phi(e) = \phi(\partial e) = 0 \Rightarrow \phi$ is constant.

$\cong \mathbb{Z}$ (constant functions).

$H^1(\mathbb{R};\mathbb{Z})$ $\quad \Delta^1(\mathbb{R};\mathbb{Z})/\mathrm{im}(\delta)$ $\quad$ given f: edges $\to \mathbb{Z}$.

construct g by $g(v_0) = 0$

$g(v_1) = f(e_1)$

$g(v_k) = f(e_1)+f(e_2)+\dots+f(e_k)$

etc.

so $H^1(\mathbb{R};\mathbb{Z}) \cong 0$.

$H^0_c(\mathbb{R};\mathbb{Z}) = 0$: as $\delta(\Delta^0_c(\mathbb{R};\mathbb{Z})) = 0$, as the only constant function with compact support is 0.

$H^1_c(\mathbb{R};\mathbb{Z}) \cong \mathbb{Z}$: $\quad \Delta^1_c(\mathbb{R};\mathbb{Z})/\mathrm{im}(\delta)$ $\quad$ consider $\Sigma : \Delta^1_c(\mathbb{R};\mathbb{Z}) \to \mathbb{Z}$

$\phi \longmapsto \sum_i \phi(e_i)$

(Σ not defined on all of Δ^1, just Δ_c). note: $\Sigma(\delta\psi) = \sum \delta\psi(e_i) = \sum \psi(\partial e_i) = 0$

so defines a map $H^1_c(\mathbb{R};\mathbb{Z}) \to \mathbb{Z}$

claim: Σ injective: suppose $\Sigma(\phi) = 0$ define $\psi(v_i) = \sum_{j=-\infty}^{i} \phi(e_j)$, has compact support (!) and $\delta\psi = \phi$.

In general: $\quad C^i(X;G)$ singular i-chains

$\cup$

$C^i_c(X;G)$ subgroup consisting of chains ϕ for which there is a compact set $K_\phi \subset X$ s.t. ϕ is zero on all chains in $X \setminus K_\phi$

$\phi : C_i(X) \to G$

note: $\delta\phi$ is also zero on all chains in $X\setminus K_\phi$, so $C^i_c(X;G)$ is a subchain complex of $C^i(X;G)$, with cohomology groups $H^i_c(X;G)$ the <u>cohomology groups with compact support</u>.

Alternate description via algebraic limits.

$$C^i_c(X;G) = \bigcup_{K \text{ compact}} C^i(X, X\setminus K; G)$$

inclusion $K \hookrightarrow L$ induces $C^i(X|K) \hookrightarrow H^i(X|L)$

gives $H^i(X|K) \to H^i(X|L)$

$\{G_\alpha \mid \alpha \in I\}$ abelian groups indexed by $\alpha \in I$ directed set, i.e. for each $\alpha, \beta \in I$ there is γ with $\alpha \le \gamma$, $\beta \le \gamma$.

if given $\alpha \le \beta$ have $f_{\alpha\beta} : G_\alpha \to G_\beta$ with $f_{\alpha\alpha} = \mathbb{1}_{G_\alpha}$, and $\alpha \le \beta \le \gamma$ implies $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$ then this is a directed system of abelian groups

can define direct limit $\varinjlim G_\alpha$

Defn $\varinjlim G_\alpha = \bigoplus_\alpha G_\alpha \Big/ \langle \underset{\in G_\alpha}{a} = f_{\alpha\beta}(a) \rangle$

Defn $\coprod_\alpha G_\alpha / \sim$ where $\underset{\in G_\alpha}{a} \sim \underset{\in G_\beta}{b}$ if $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ for some γ.

Check: two definitions are equivalent and define an abelian group.

Application $X = \bigcup X_\alpha$ X_α directed set under inclusions

gives $H_i(X_\alpha; G) \to H_i(X; G)$ for each α, defines $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$

Propn If $X = \bigcup X_\alpha$ directed set under inclusion, s.t. every compact set is contained in some X_α, then the natural map $\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$ is an isomorphism for all i and G.

Proof surjective: represent $[z] \in H_i(X; G)$ by cycle z, compact, contained in some X_α. injective: if $z = \partial w$, then w compact, lies in some $X_{\alpha'}$. $\square$

$K \subset X$ compact — compact sets form a direct set under inclusion.

$K \subset L$ get $H^i(X|K) \to H^i(X|L)$ and $\varinjlim H^i(X|K) = H^i_c(X)$

<u>Example</u> $H^*_c(\mathbb{R}^n; G) = \begin{cases} G & i=n \\ 0 & i \neq n \end{cases}$

can choose $K = B_k = \{x \in \mathbb{R}^n \mid \|x\| \leq k\}$.
but then $H^i(\mathbb{R}^n|0) = \begin{cases} G & i=n \\ 0 & i\neq n \end{cases}$ and $H^i(\mathbb{R}^n|B_k) \xrightarrow[\text{iso}]{\cong} H^i(\mathbb{R}^n|B_{k+1})$ $\square$.

<u>Warning</u> $H^*_c(\mathbb{R}^n; G)$ <u>not</u> an invariant of homotopy type!

[$f: X \to Y$ f^{-1}(compact) need not be compact].

<u>Duality for non-compact manifolds</u>

define $D_M : H^k_c(M;R) \to H_{n-k}(M;R)$ $\qquad K \subset L \subset M$ compact

$$\begin{array}{ccc} \mu_L \quad H_n(M|L) \times H^k(M|L) & \xrightarrow{\frown} & \\ \downarrow i_* \qquad \uparrow i^* & & H_{n-k}(M;R) \\ \mu_K \quad H_n(M|K) \times H^k(M|K) & \xrightarrow{\frown} & \end{array}$$

M orientable gives unique μ_K, μ_L from orientation class, so $\mu_K \overset{i_*}{\longmapsto} \mu_L$

naturality: $i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$ for $x \in H^k(M|K)$

so $\mu_K \frown x = \mu_L \frown i^*(x)$

so let K vary over compact sets get $D_M : \varinjlim H^k(M|K) \to H_{n-k}(M)$
$\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad H^k_c(M)$.

<u>Thm</u> $D_M : H^k_c(M;R) \to H_{n-k}(M)$ is an isomorphism for all k, whenever M is an R-oriented manifold.