Proof (of Lemma).

1) if true for $A, B, A\cap B$ compact, then true for $A\cup B$.

Mayer–Vietoris sequence:

$$\cdots \to \underset{H_{n+1}(M|A\cap B)}{0} \to H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A)\oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A\cap B) \to \cdots$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$(\alpha, \beta) \longmapsto \alpha+\beta.$$

$0 \to H_{n+i}(M|A\cup B) \to 0 \Rightarrow H_{n+i}(M|A\cup B) = 0$ for $i>0$. (b)).

spose $x \mapsto \alpha_x$ is a section, then get unique α_A α_B $\alpha_{A\cap B}$.

$$(\alpha_A, -\alpha_B) \longmapsto \underset{\alpha_{A\cap B}}{\alpha_A} - \underset{\alpha_{A\cap B}}{\alpha_B} = 0$$

So exists $\alpha_{A\cup B}$ s.t. $\alpha_{A\cup B} \mapsto (\alpha_A, -\alpha_B)$.

so $\alpha_{A\cup B} \mapsto \alpha_x$ for all $x\in A\cup B$ as $\alpha_A \mapsto \alpha_x$ for all $x\in A$ and $\alpha_B \mapsto \alpha_x$ for all $x \in B$.

$\alpha_{A\cup B}$ unique as Φ injective.

2) Reduce to case $M=\mathbb{R}^n$.

Any compact $A\subset M$ can be written as a finite union of compact sets $A_1 \dots A_k$, each contained in an open $\mathbb{R}^n \subset M$.

use 1) to split down to 1-compact set contained in $\mathbb{R}^n$ and apply excision.

3) $M=\mathbb{R}^n$ $A\subset M$ convex then $H_i(\mathbb{R}^n|A) \to H_i(\mathbb{R}^n|x)$ is an isomorphism for any $x\in A$, as both $\mathbb{R}^n\setminus A$ and $\mathbb{R}^n\setminus x$ deformation retract onto a ~~small~~ sphere centered at x.

Works for any finite union of convex sets using 2), 1).

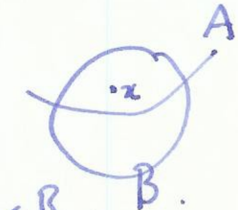
4) $A\subset\mathbb{R}^n$ compact. Let $\alpha\in H_i(\mathbb{R}^n|A)$ be represented by a relative cycle z and let $C=$ image of ∂z, also compact, and has positive distance δ from A. Cover A by finitely many closed balls of radius $<\delta$ centered at

points of A, let $K =$ union of these balls, disjoint from C

Z defines a relative cycle for $\alpha_K \in H_i(\mathbb{R}^n | K)$

K union of convex sets, so $H_i(\mathbb{R}^n|K) = 0$, $i > n$, by 3).

so $\alpha_K = 0 \Rightarrow \alpha = 0$. so $H_i(\mathbb{R}^n|A) = 0$.



$i=n$: if $\alpha_x = 0$ in $H_n(\mathbb{R}^n|x)$ for all $x \in A$ then
$\alpha_x = 0$ in $H_n(\mathbb{R}^n|x)$ for all $x \in K$

as $H_n(\mathbb{R}^n|B) \to H_n(\mathbb{R}^n|x)$ is an isomorphism for all $x \in B$.

so $\alpha_x = 0$ for all $x \in K \Rightarrow \alpha_K = 0 \Rightarrow \alpha = 0$ this gives uniqueness in Thm a).

existence: just let α_A be the image of α_B for some ball B with $A \subset B$. □

Non-compact case

Prop^n $H_i(M;R) = 0$ for $i \geq n$

Proof represent $H_i(M;R)$ by a cycle z. This has compact image in M so there is an open set $U \subset M$ containing image of z, with compact closure $\bar{U} \subset M$. Set $V = M \setminus \bar{U}$. L.e.s. of triple $(M, U\cup V, V)$

$$\begin{array}{ccccc} H_{i+1}(M, U\cup V; R) & \longrightarrow & H_i(U\cup V, V; R) & \longrightarrow & H_i(M, V; R) \; ② \\ ① & & \uparrow & & \uparrow \\ & & H_i(U;R) & \longrightarrow & H_i(M;R) \end{array}$$

for $i > n$ ① ② $= 0$ by lemma $\Rightarrow$ middle group 0.

$i=n$ $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ of $M_R \to M$
M connected, section determined by value at a single point so $z = 0$ if $[z]_x = 0$ for some x, but just choose $x \notin \text{image}(z)$, then $[z]_x = 0$ ⊠
but $[z]$ any element of $H_n(M;R) \Rightarrow H_n(M;R) = 0$ □.

Thm (Poincaré duality) M closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D: H^k(M;R) \to$ ~~H_{n-k}~~ $H_{n-k}(M;R)$ given by $D(\alpha) = [M] \cap \alpha$ is an isomorphism for all k.

Cap product $\quad \cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \qquad (k \geq \ell)$

$\sigma : \Delta^k \to X$
$\phi \in C^\ell(X;R)$

$$\sigma \cap \phi = \phi\left(\sigma|_{[v_0,\dots,v_\ell]}\right)\sigma|_{[v_\ell,\dots,v_k]}$$

(bilinear on $C_k \times C^\ell$)

this gives a map on $\quad H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k-\ell}(X;R)$

as $\quad \partial(\sigma \cap \phi) = (-1)^\ell\left(\partial\sigma \cap \phi - \sigma \cap \delta\phi\right)$

check:

$$\partial\sigma \cap \phi = \sum_{i=0}^{\ell} (-1)^i \phi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]}\right)\sigma|_{[v_{\ell+1},\dots,v_k]}$$
$$+ \sum_{i=\ell+1}^{k} (-1)^i \phi\left(\sigma|_{[v_0,\dots,v_\ell]}\right)\sigma|_{[v_\ell,\dots,\hat{v}_i,\dots v_k]}$$

$$\sigma \cap \delta\phi = \sum_{i=0}^{\ell+1} (-1)^i \phi\left(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]}\right)\sigma|_{[v_{\ell+1},\dots,v_k]}$$

$$\partial(\sigma \cap \phi) = \sum_{i=\ell}^{k} (-1)^{i-\ell} \phi\left(\sigma|_{[v_0,\dots v_\ell]}\right)\sigma|_{[v_\ell,\dots \hat{v}_i,\dots v_k]}.$$

$\Rightarrow \quad$ cycle $\cap$ cocycle is a cycle.

boundary $\cap$ cocycle is a boundary: $\quad \sigma = \partial\alpha,\ \delta\phi = 0 \qquad \partial(\alpha \cap \phi) = \pm\, \partial\alpha \cap \phi = \pm\, \sigma \cap \phi.$

cycle $\cap$ coboundary is a boundary

$\partial\sigma = 0,\ \delta\psi = \phi \qquad \partial(\sigma \cap \psi) = \pm\, \sigma \cap \delta\psi = \pm\, \sigma \cap \phi.$

this gives $\quad H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$

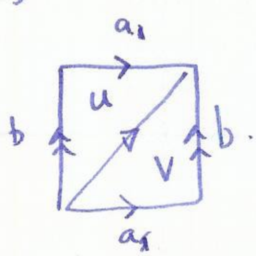
relative versions:
$$H_k(X,A;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$
$$H_k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$
$$H_k(X,A\cup B;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X,B;R)$$

$f: X \to Y$
$$\begin{tikzcd} \alpha \quad f^*(\phi) & & \alpha \frown f^*(\phi) \\ H_k(X) \times H^\ell(X) \arrow[r, "\frown"] \arrow[d] & H_{k-\ell}(X) \arrow[d] \\ H_k(Y) \times H^\ell(Y) \arrow[r, "\frown"] \arrow[u] & H_{k-\ell}(Y) \\ f_*(\alpha) \quad \phi \end{tikzcd}$$

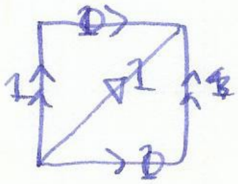
$$f_*(\alpha) \frown \phi = f_*(\alpha \frown f^*(\phi))$$

<u>Thm</u> (Poincaré duality) If M is a closed R-orientable n-manifold, with fundamental class $[M] \in H_n(M;R)$, then the map $D: H^k(M;R) \to H_{n-k}(M;R)$ given by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.

<u>Example</u>

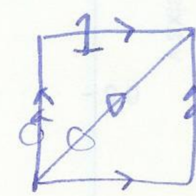


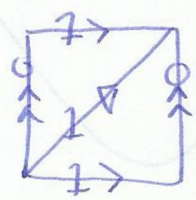
$[M] = u - v$



α.

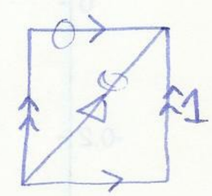
$[M] \frown \alpha$





β

$[M] \frown \beta$



recall: M closed means compact.
if M open, need cohomology with compact supports.

<u>simplicial cohomology with compact supports.</u>

X Δ-complex, locally compact $\quad \Delta^i_c(X;G) < \Delta^i(X;G)$ subgroup of compactly supported cochains, i.e. non-zero on only finitely many i-simplices.
$\delta\phi$ also supported on only finitely many simplices, so $\Delta^i_c(X;G)$ is a ~~subcomplex~~ chain of the simplicial cohomology chain, with cohomology groups $H^i_c(X;G)$.