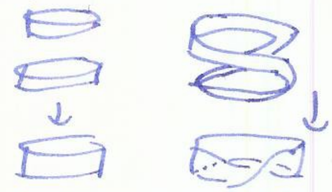
$H_n(B|x) \cong H_n(U(\mu_B)|\mu_x) \cong H_n(\tilde{M}|\mu_x)$, and satisfies local consistency by defn. □.

Propn: If M is connected then M is orientable iff $\tilde{M}$ has two components.
In particular, M is orientable if $\pi_1 M$ has no subgroup of index 2.
(eg $\pi_1 M$ trivial).



Proof if M connected ~~orientable~~, then $\tilde{M}$ has either one or two components. If two components, then each mapped homeomorphically to M, and they correspond to the two possible orientations on M. □

Generalization $H_n(M|x;R) \cong R$. An R-orientation assigns to each $x \in M$ a generator μ_x of $H_n(M|x;R)$ (generator of R is an element $\overset{a}{\mu_x}$ s.t. $aR = R$), subject to the local consistency condition.

Fact: every manifold is $\mathbb{Z}_2$-orientable. $(H_n(M|x;\mathbb{Z}_2) \cong \mathbb{Z}_2)$.

Thm M closed n-manifold.

a) If M is orientable, the map $H_n(M;\mathbb{Z}) \to H_n(M|x;\mathbb{Z}) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$.
[For all M, $H_n(M;\mathbb{Z}_2) \to H_n(M|x;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is an isomorphism].

b) If M is not orientable, $H_n(M;\mathbb{Z})$ ~~$\to H_n(M|x;\mathbb{Z})$~~ $\cong 0$.
$\to H_n(M|x;\mathbb{Z})$ injective with image $2a = 0$.

c) $H_i(M;\underset{\mathbb{Z}_2}{\mathbb{Z}}) = 0$ for all $i > n$.

In particular $H_n(M;\mathbb{Z}) = \mathbb{Z}$ orientable, 0 un-orientable; $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Defn An element of $H_n(M;\mathbb{Z}_{(2)})$ whose image in $H_n(M|x)$ is a generator for all x is called a fundamental class for M, or an orientation class.

Special case M^n is a Δ-complex.
$\mathbb{Z}_2$ coeffs: Let $[M]$ be one copy of each n-simplex. To be a manifold, each face occurs in exactly two n-simplices, so $\partial M = 0$ (mod 2).

$\mathbb{Z}$ coeffs: let $M = \sum_{\sigma_i} k_i \sigma_i$ $\quad k_i = \pm 1$.

$\partial M = 0$ if you can choose an appropriate set of k_i, can do this iff M^n orientable.

Corollary M closed connected n-manifold, then $tor(H_{n-1}(M;\mathbb{Z})) = 0$ orientable

$= \mathbb{Z}_2$ non-orientable.

Proof universal coefficients for homology (proof postponed) (§3.4)

$$0 \to H_n(M;\mathbb{Z}) \otimes G \to H_n(M;G) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), G) \to 0$$

split exact

How to compute Tor:

1) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$
2) $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$
3) $\mathrm{Tor}(A,B) = 0$ if A or B torsion free
4) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(T(A),B)$ $\quad T(A)$ torsion subgroup of A.
5) $\mathrm{Tor}(\mathbb{Z}_n, A) \cong \ker(A \xrightarrow{n} A)$

Orientable case: $0 \to \mathbb{Z} \otimes \mathbb{Z}_p \to H_n(M;\mathbb{Z}_p) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}_p) \to 0$

$\mathbb{Z}_p \qquad \mathbb{Z}_p \qquad \Rightarrow 0$

Non-orientable case: $0 \to 0 \to H_n(M;\mathbb{Z}_p) \to \mathrm{Tor}(H_{n-1}(M;\mathbb{Z}), \mathbb{Z}_p) \to 0$

$p=2 \quad \mathbb{Z}_2 \qquad \Rightarrow \mathbb{Z}_2. \Rightarrow H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}_2$

$p \neq 0 \quad 0 \qquad \Rightarrow 0$

$\square$

special case M^n cell structure with 1 n-cell.

cellular chain complex $\quad 0 \to \underset{\mathbb{Z}}{C_n(M)} \xrightarrow{d_n} C_{n-1}(M) \to C_{n-2}(M) \to \cdots$

M orientable $\Rightarrow H_n(M) \cong \mathbb{Z} \Rightarrow d_n = 0 \Rightarrow C_{n-1}(M)$ free (ie. no torsion).

M non-orientable: $H_n(M;\mathbb{Z}_p) = 0 \quad p \neq 2$

$\mathbb{Z}_2 \quad p = 2$

$0 \to C_n(M) \xrightarrow{d_n} C_{n-1}(M) \to C_{n-2}(M)$

$\mathbb{Z} \qquad \mathbb{Z} \oplus \mathbb{Z}^k$

$1 \longmapsto 2$(generator) of a $\mathbb{Z}$ summand.

$(2, 0, \dots, 0)$

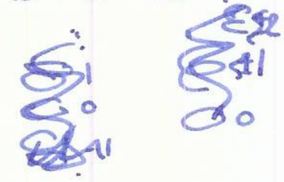
so $H_{n-1}(M;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^l$.

$\mathbb{Z}$ coeffs: can generalize $\tilde{M} \to M$ to $M_{\mathbb{Z}} \to M$

$M_{\mathbb{Z}} =$ all $\{\alpha_x \in H_n(M|x),\ x \in M\}$

topology: $U(\alpha_B) = \{ \alpha_x \mid x \in B$ and $H_n(M|B) \to H_n(M|x)$ $\}$. B ball $\subset \mathbb{R}^n$

$\alpha_B \longmapsto \alpha_x$



structure of $M_{\mathbb{Z}}$: $\alpha_x = 0$: just get $M_0 \cong M$. orientable:

$\alpha_x = k$ generator get $M_k \cong \tilde{M}$. $k = 1, 2, \dots$

$\{\pm k\alpha_x\}$ at each point.

<u>Def</u>

Def^{n} A cts map $M \to M_{\mathbb{Z}}$ is a <u>section</u> of $M_{\mathbb{Z}}$ (ie. $p(\alpha_x) = x$).

$x \mapsto \alpha_x$

An <u>orientation</u> of M is a section s.t. α_x is a generator of $H_n(M|x)$ for each x.

structure of $M_{\mathbb{Z}_2}$: $\alpha_x = 0$ just get $M_0 \cong M$

$\alpha_x = 1$ just get $\tilde{M}_1 = M$

<u>Lemma</u> M^n, $A \subset M$ compact subset. Then

a) If $x \mapsto \alpha_x$ is a section of $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M|A; R)$ whose image in $H_n(M|x; R)$ is α_x for all $x \in A$.

b) $H_i(M|A; R) = 0$ for $i > n$.

Lemma $\Rightarrow$ theorem: c) $H_i(M; \mathbb{Z}) = 0$ for $i > n$ (choose $A = M$).
 $\mathbb{Z}_2$

for a), b): let $\Gamma_R(M)$ be the set of sections of $M_R \to M$

$\uparrow$ R-module! can add sections, and multiply by R.

there is a homomorphism $H_n(M; R) \to \Gamma_R(M)$

$\alpha \mapsto (x \mapsto \alpha_x)$.

$\qquad \| $

$H_n(M; R) \to H_n(M|x; R)$

$\alpha \longmapsto \alpha_x$.

Lemma.

a) $\Rightarrow$ homomorphism is an isomorphism.

M non-orientable: this map must be zero as $\Gamma_{\mathbb{Z}}(M) \sim M_0 \cup$ double cover.

M orientable, ~~α_x must map to generator, two components~~. $\Gamma_{\mathbb{Z}}(M) \cong \mathbb{Z}$. $\square$.