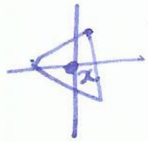
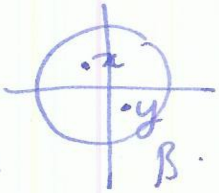
Def^n An orientation of $\mathbb{R}^n$ at x is a choice of generator for $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; \mathbb{Z})$

recall:  $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; \mathbb{Z}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{Z}$

rotations send $\alpha \mapsto \alpha$
reflections send $\alpha \mapsto -\alpha$

note: $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$



so a choice of generator at one point of $\mathbb{R}^n$ determines a choice of generator at all points of $\mathbb{R}^n$.

Def^n A local orientation at $x \in M^n$ is a choice of generator for $H^n(M^n, M^n \setminus \{x\})$

notation write $H^n(X|A)$ for $H^n(X, X \setminus A)$ ← only depends on a small nbd of A in X.

global orientation: "consistent choice of local orientation".

Def^n An orientation of M^n is a function $\mu : x \mapsto \mu_x$ sending $x \in M$ to a generator μ_x of $H_n(M|x)$ (i.e. a local orientation) s.t. for each $x \in M$ there is an open nbd $B \cong \mathbb{R}^n$, and a local orientation μ_B s.t. for all $y \in B$, the map $H_n(M|B) \to H_n(M|y)$ takes μ_B to μ_y.

If M has an orientation then we say M is orientable.

Some manifolds are not orientable. Examples: $\mathbb{RP}^2$, Klein bottle.

Prop^n Every manifold M has an orientable two-fold cover $\tilde{M}$

Proof: Set $\tilde{M} = \{\mu_x \mid x \in M, \mu_x \text{ is a local orientation of } M \text{ at } x\}$ as a set.

then $\underset{\mu_x \mapsto x}{\tilde{M} \to M}$. Topology on $\tilde{M}$: let $\underset{\text{open ball}}{B} \subset \mathbb{R}^n \subset M^n$, set

$U(B) = \{\mu_x \mid x \in B \text{ and the map } H_n(M|B) \to H_n(M|x) \text{ takes } \mu_B \text{ to } \mu_x\}$.

Exercise: check this is a base for a topology on $\tilde{M}$, and that $\tilde{M} \to M$ is a covering map.

note: $\tilde{M}$ is orientable, as $\mu_x \in \tilde{M}$ has a canonical local orientation given by