§3.3 Poincaré duality

Def An n-manifold M is a topological space in which every point has an open neighbourhood homeomorphic to $\mathbb{R}^n$, Hausdorff, 2nd countable.

Note: local homology groups

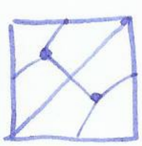
$$H_i(M, M\setminus\{x\}; \mathbb{Z}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \quad \text{(excision)}$$
$$\cong \tilde{H}_{i-1}(\mathbb{R}^n\setminus\{0\}) \quad \text{(l.e.s of pair)}$$
$$\cong \tilde{H}_{i-1}(S^{n-1})$$
$$\cong \mathbb{Z} \text{ in dim } n$$
$$0 \text{ otherwise.}$$

Def A compact n-manifold is called closed.

Examples S^n, $\mathbb{R}^n$, $\mathbb{RP}^n$, surfaces S_g, lens spaces, T^n

Poincaré duality (M^n orientable): $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$ ($\cong H_{n-k}(M;\mathbb{Z})$ modulo torsion)

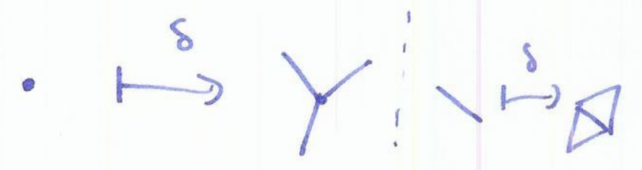
intuition: dual cell structures ($\mathbb{Z}_2$-coeffs)



$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \qquad \partial: \sigma \longmapsto \sum_{i=0}^{n} (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]$

identify the dual $\mathrm{Hom}(C_2, \mathbb{Z}_2)$ with the dual cell group C_0^*

$$C_0^* \xrightarrow{\delta} C_1^* \xrightarrow{\delta} C_2^*$$



$$\delta(\varphi(\sigma)) = \varphi(\partial\sigma)$$

so we get

$$\begin{array}{ccccc} C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 \\ \Big\downarrow\wr & & \Big\downarrow\wr & & \Big\downarrow\wr \\ C_0^* & \xrightarrow{\delta} & C_1^* & \xrightarrow{\delta} & C_2^* \end{array}$$

commutes

to get this to work with $\mathbb{Z}$-coefficients need a coherent choice of signs for each identification: i.e. an orientation.