$$= \epsilon_{n-1} \sum_{i=0}^{n} (-1)^{n-i} \sigma\big|_{[v_n, \dots, \hat{v}_{n-i}, \dots, v_0]}$$

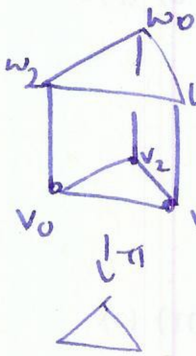
so $\partial\rho(\sigma) = \rho\partial(\sigma) \iff \epsilon_n = (-1)^n \epsilon_{n-1}$ (exercise).

<u>chain homotopy</u>

$$\begin{tikzcd} \leftarrow C^k(X;R) \arrow[d, "\rho"] & C^{k-1}(X;R) \arrow[l, "\delta"'] \arrow[d, "\rho"] & \leftarrow \cdots \\ \cdots \leftarrow C^k(X;R) & C^{k-1}(X;R) \arrow[l, "\delta"'] & \leftarrow \cdots \end{tikzcd}$$

$P : C_n(X) \to C_{n+1}(X)$

$$\sigma \longmapsto (-1)^i \epsilon_{n-i} (\sigma\pi)\big|_{[v_0, \dots, v_i, w_n, \dots, w_i]}$$



this is the old prism operator P, but with top vertices w_i written in reverse order.

<u>claim</u> $\partial P + P\partial = \rho - \mathbb{1}$. $\square$

<u>A Künneth formula</u>

recall: cross product
$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$
$$a \times b \longmapsto p_1^*(a) \cup p_2^*(b)$$

this map is bilinear, so not a homomorphism in general.

<u>Tensor products</u> (of abelian groups)

A, B abelian groups

$A \otimes B$ abelian group: generators $a \otimes b$ $\quad a \in A, b \in B$

relations $(a+a') \otimes b = a\otimes b + a' \otimes b$

$a \otimes (b+b') = a\otimes b + a\otimes b'$

zero: $0 \otimes 0 = 0 \otimes b = a \otimes 0$

inverse: $-(a\otimes b) = (-a)\otimes b = a \otimes (-b)$

useful facts: $A \otimes B \cong B \otimes A$

$$\bigoplus_i A_i \otimes B \cong \bigoplus (A_i \otimes B)$$

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

$$\mathbb{Z} \otimes A \cong A$$

$$\mathbb{Z}_n \otimes A \cong A/nA$$

· homomorphisms $f: A \to A'$, $g: B \to B'$ induce a homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$

$$f \otimes g : (a \otimes b) \mapsto f(a) \otimes g(b)$$

· a bilinear map $\phi : A \times B \longrightarrow C$ induces a homomorphism $A \otimes B \longrightarrow C$

$$a \otimes b \to \phi(a, b)$$

(modules over a commutative ring R)

$A \otimes_R B = A \otimes B$ generators: $a \otimes b$

relations: $(a + a') \otimes b = a \otimes b + a' \otimes b$ $\quad a, a' \in A$, $b, b' \in B$.

$a \otimes (b + b') = a \otimes b + a \otimes b'$

and: $ra \otimes b = a \otimes rb$ $\quad r \in R$.

then $A \otimes_R B$ is an R-module.

Note $A \otimes_{\mathbb{Q}} B = A \otimes B = A \otimes_{\mathbb{Z}_n} B$ but not true for general rings.

Example $R = \mathbb{Q}(\sqrt{2})$ (2-d vector space over $\mathbb{Q}$) so $R \otimes R$ 4-d vector space over $\mathbb{Q}$

but $R \otimes_R R = R$.

Cross product the bilinear map

$$H^*(X;R) \times H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R)$$

$$a \quad\quad b \quad\quad\quad p_1^*(a) \cup p_2^*(b)$$

gives rise to a homomorphism

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y; R)$$

$$a \otimes b \longmapsto a \times b.$$

this is a ring homomorphism if we set $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$

check: $(-1)^{|b||c|}\, ac \times bd = (-1)^{|b||c|}\, p_1^*(a \cup c) \cup p_2^*(b \cup d)$

$$= (-1)^{|b||c|}\, p_1^*(a) \cup p_1^*(c) \cup p_2^*(b) \cup p_2^*(d)$$

$$= p_1^*(a) \cup p_2^*(b) \cup p_1^*(c) \cup p_2^*(d)$$

$$= (a \times b)(c \times d).$$

Thm (simple Künneth) The cross product $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y; R)$ is ~~an~~ ring isomorphism if X, Y CW-complexes and $H^k(Y;R)$ finitely generated free R-module for all k.

Example $H^*(\mathbb{RP}^\infty \times \mathbb{RP}^\infty; \mathbb{Z}_2) \cong H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \otimes H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha] \otimes \mathbb{Z}_2[\beta] = \mathbb{Z}_2[\alpha, \beta]$.

$H^*(S^1 \times \cdots \times S^1; \mathbb{Z}) = \Lambda_R[\alpha_1, \ldots, \alpha_n]$ exterior algebra on $\alpha_1, \ldots, \alpha_n$.

Proof fix Y, and consider functors:

$$h^n(X,A) = \bigoplus_i \left(H^i(X,A;R) \otimes_R H^{n-i}(Y;R) \right)$$

$$k^n(X,A) = H^n(X \times Y, A \times Y; R).$$

natural transformation: $\mu: h^n(X,A) \to k^n(X,A)$

$$a \otimes b \longmapsto a \cup b$$

claim: h^n, k^n are cohomology theories.

Axioms: 1) homotopy invariance $f \simeq g \Rightarrow f^* = g^*$. ✓.

2) excision: $h^*(X,A) \cong h^*(B, A \cap B)$ [A, B subcomplexes and $X = A \cup B$].

k^*: $(A \times Y) \cup (B \times Y) = (A \cup B) \times Y$

$(A \times Y) \cap (B \times Y) = (A \cap B) \times Y$.

3) long exact sequence of a pair: k^*: (short exact sequence works) ✓.

h^*: take l.e.s of (X,A):

$$\cdots \leftarrow H^{n+1}(X,A) \leftarrow H^n(A) \leftarrow H^n(X) \leftarrow H^n(X,A) \leftarrow \cdots$$

tensor with $H^n(Y;R)$ (free R-module so still get exact sequence)
(direct sum of copies of R)

now let n vary and take direct sums of exact sequences, shifted to get correct dimensions.

4) disjoint unions: k^* ✓

h^*: algebraic fact: M_α R-module, N f.g. free R-module

$$\left(\coprod_\alpha M_\alpha\right)\otimes_R N \cong \coprod_\alpha \left(M_\alpha \otimes_R N\right)$$

as $N = \coprod_\beta R_\beta$, so $M_\alpha \otimes_R N \cong \coprod_\beta M_\alpha \otimes_R N_\beta$

$$\coprod_\beta \coprod_\alpha M_\alpha \otimes R_\beta = \coprod_\alpha \coprod_\beta M_\alpha \otimes N_\beta .$$

<u>claim</u>: μ is a natural transformation

$f: X \to Y$ $\quad \mu(f): h^n(X) \to k^n(X)$

$a\otimes b \longmapsto a\cup b$ $\quad$ cup product is natural.

coboundary maps in l.e.s of a pair:

$$\begin{tikzcd}
H^{k+1}(X,A;R)\times H^\ell(Y;R) \arrow[d] & H^k(A;R)\times H^\ell(Y;R) \arrow[l, "\delta\times 1"'] \arrow[d] \\
H^{k+\ell+1}(X\times Y, A\times Y;R) & H^{k+\ell}(A\times Y;R) \arrow[l, "\delta"']
\end{tikzcd}$$

$\phi \in C^k(A;R)$ extended to $\bar\phi \in C^k(X;R)$

$\psi \in C^\ell(Y;R)$

$$\begin{tikzcd}
(\delta\bar\phi, \psi) \arrow[d, mapsto] & (\phi,\psi) \arrow[l, mapsto] \arrow[d, mapsto] \\
p_1^\#(\delta\bar\phi)\cup p_2^\#(\psi) & p_1^\#(\phi)\cup p_2^\#(\psi) \arrow[dl, mapsto] \\
\delta\left(p_1^\#(\bar\phi)\cup p_2^\#(\psi)\right) &
\end{tikzcd}$$

$= \delta p_1^\#\bar\phi \cup p_2^\#(\psi) + (-1)^k p_1^\#(\phi)\cup \delta p_2^\#(\psi)$

$= p_1^\#(\delta\bar\phi)\cup p_2^\#(\psi) + (-1)^k p_1^\#(\phi)\cup p_2^\#(\underset{0}{\underset{\shortparallel}{\delta\psi}})$ ✓.

<u>summary</u>: h^*, k^* are cohomology theories, μ is a natural transformation between them.

<u>Prop^n</u> If a natural transformation between unreduced cohomology theories of CW-pairs is an isomorphism for (point, ϕ), then it is an isomorphism for all pairs.

note: $h^*(\{pt\},\emptyset) = \bigoplus_i H^i(\{pt\},\emptyset;R)\otimes_R H^{n-i}(Y;R) = H^n(Y;R)$

$k^*(\{pt\},\emptyset) = H^n(Y,\emptyset;R)$

so this implies $h^*(X,A) \cong k^*(X,A)$ for all CW-pairs (X,A).

Proof (of Prop^n) Since $\mu: h^*(X,A) \to k^*(X,A)$ is natural..

assume X finite dimensional, do induction on dimension.

$\dim(X) = 0$, holds by hypothesis and disjoint union axiom.

induction step: long exact sequence for (X^n, X^{n-1}) gives:

$$\begin{tikzcd} \cdots & h^{m+1}(X^n,X^{n-1}) \arrow[l] \arrow[d, "\mu"] & h^m(X^{n-1}) \arrow[l] \arrow[d, "\mu"] & h^m(X^n) \arrow[l] \arrow[d, "\mu"] & h^m(X^n,X^{n-1}) \arrow[l] \arrow[d, "\mu"] & \cdots \arrow[l] \\ \cdots & h^{m+1}(X^n,X^{n-1}) \arrow[l] & k^m(X^{n-1}) \arrow[l] & k^m(X^n) \arrow[l] & k^m(X^n,X^{n-1}) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

commutes (μ natural).

Five lemma: suffices to show μ is an iso for $(X,A) = (X^n, X^{n-1})$

attaching maps: $\Phi: \coprod_\alpha (D^n_\alpha, \partial D^n_\alpha) \to (X^n, X^{n-1})$ i.e. $\phi_\alpha: \partial D^n_\alpha \to X^{n-1}$.

excision + disjoint unions means suffices to show for a single pair $(D^n, \partial D^n)$

l.e.s: $\cdots \leftarrow h^{m+1}(D^n,\partial D^n) \leftarrow h^m(\partial D^n) \leftarrow h^m(D^n) \leftarrow h^m(D^n,\partial D^n) \leftarrow \cdots$

$\downarrow$ $\quad\downarrow$ induction $\quad\downarrow$ contractible $\quad$! .

0.

5 lemma $\Rightarrow$ works for $D^n, \partial D^n$. $\square$.

General Künneth

Th^m X, Y CW-complexes, R principal ideal domain, then

$$0 \to \bigoplus_i \left(H_i(X;R)\otimes_R H_{n-i}(Y;R)\right) \to H_n(X\times Y;R) \to \bigoplus_i \mathrm{Tor}_R\left(H_i(X;R), H_{n-i-1}(Y;R)\right) \to 0$$

short exact, split (not natural splitting).

Relative Künneth

$H^*(X,A;R)\otimes_R H^*(Y,B;R) \to H^*(X\times Y, A\times Y \cup X\times B; R)$ is an iso if

$H^k(Y,B;R)$ f.g. free R-module for each k.

Thm (Hopf) If $\mathbb{R}^n$ has a division algebra structure over $\mathbb{R}$, then n is a power of 2.

Recall ~~division~~ algebra. $(a,b) \to ab$ bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

so $a(b+c) = ab+ac$ $\quad \alpha(ab) = (\alpha a)b = a(\alpha b)$ $\alpha \in \mathbb{R}$. not (nec) commutative
$(a+b)c = ac+bc$ associative
with identity element.

division algebra: $ax=b$ always solvable whenever $a \neq 0$.
$x = a \div b$ i.e. no zero-divisors, i.e. $\mathbb{R}^n \to \mathbb{R}^n$
$x \mapsto ax$ surjective.
$x \mapsto xa$

Examples $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. that's all [Bott-Milnor] [Kervaire]
1 2 4 8

Proof (Hopf) $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ gives $g: S^{n-1} \times S^{n-1} \to S^{n-1}$
$$(x,y) \mapsto \frac{xy}{|xy|}$$

note. $-(xy) = (-x)y = x(-y)$
so $g(-x,y) = -g(x,y) = g(x,-y)$ so get $h: \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$
$$h^*: H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \to H^1(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}_2)$$
claim: $\gamma \longmapsto \alpha + \beta$

where γ generates $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2$ and $\alpha = p_1^*(\gamma)$, $\beta = p_2^*(\gamma)$.
$H^1 \cong \mathrm{Hom}(H_1)$, $H_1 \cong \mathrm{ab}(\pi_1) = \mathbb{Z}_2$.
$\gamma \in \pi_1 \mathbb{R}P^{n-1} \leftrightarrow$ path $\lambda: I \to S^{n-1}$ s.t. $\lambda(0) = -\lambda(1)$
fix y: $g(\lambda(t), y) = \lambda(t)y$ so $g(\lambda(0), y) = g(-\lambda(1), y) = -g(\lambda(1), y)$
so $g(\lambda(t), y)$ is non-trivial in $\pi_1 \mathbb{R}P^{n-1}$. similarly for $g(x, \lambda(t))$. $\square$ claim.

recall $\gamma^n = 0 \Rightarrow 0 = h^*(\gamma^n) = (\alpha+\beta)^n = \sum_k \binom{n}{k} \alpha^k \beta^{n-k}$
equation in $H^*(\mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta]/\langle \alpha^n, \beta^n \rangle \Rightarrow \binom{n}{k} \equiv 0 \pmod 2$
for all $0 < k < n$.

claim this only happens if $n = 2^m$ for some $m \in \mathbb{N}$.

proof $(1+x)^2 = 1+2x+x^2 = 1+x^2 \pmod 2$, so $(1+x)^{2^n} = 1+x^{2^n}$
write n as a sum of powers of 2: $n = 2^{n_1} + 2^{n_2} + \dots + 2^{n_k}$ $\quad n_i < n_{i+1}$
e.g. $(1+x^2)(1+x^4) = 1+x^2+x^4+x^6 \neq 1+x^n$. $\square$