Fact $H^k(T^n;R)$ is a free R-module with basis $\alpha_{i_1} \cup \cdots \cup \alpha_{i_k}$ for $i_1 < \cdots < i_k$, where $\alpha_i \in H^1(T^n;R)$ is $p_i^*(\alpha)$ for α a generator of $H^1(S^1;R)$ and p_i is projection onto i-th factor.

Proof postponed $\square$.

other examples of explicit ring structures

Thm

$$H^*(\mathbb{R}P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1} \qquad |\alpha| = 1.$$

$$H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha].$$

$$H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\alpha^{n+1} \qquad |\alpha| = 2$$

$$H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[\alpha].$$

Recall $f: X \to Y$ induces a ring homomorphism $f^*: H^*(Y) \to H^*(X)$ so the group isomorphism $H^*(\coprod_\alpha X_\alpha;R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha;R)$ induced by inclusion maps $i_\alpha: X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$ is a ring ~~homomorphism~~ iso with respect to coordinatewise multiplication in the product.

Similarly: $\tilde{H}^*(\bigvee_\alpha X_\alpha) \cong \prod_\alpha \tilde{H}^*(X_\alpha;R)$ is a ring isomorphism (as long as (X_α, x_α) are good pairs for each α)

Example $\mathbb{C}P^2 \not\simeq S^2 \vee S^4$

	0	1	2	3	4	
$H_*(X) =$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	
$\mathbb{C}P^2$	1		α		α^2	
$S^2 \vee S^4$	1		α		β.	$(\alpha^2 = 0)$.

Thm $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$ for all $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$. (R commutative)

In particular, if $\alpha \in H^k(X)$, k odd, then $2\alpha^2 = 0 \in H^{2k}(X)$.

Proof $\phi \in C^k(X;R)$ $\psi \in C^l(X;R)$

$\phi \cup \psi$, $\psi \cup \phi$ differ by a permutation of vertices of Δ^{k+l}.

$\sigma : [v_0, \dots, v_n] \to X$, define $\bar{\sigma} = \sigma \circ \rho$ ρ linear map $\Delta^{k+l} \to \Delta^{k+l}$

which reverses the order of the vertices, i.e. $\rho(v_i) = v_{n-i}$.

in particular $\bar{\sigma}(v_i) = \sigma(v_{n-i})$.

reversing the order requires $n + (n-1) + \dots + 1 = \frac{1}{2}n(n+1)$ transpositions/reflections

so should need a sign of $\epsilon_n = (-1)^{n(n+1)/2}$.

define $\rho : C_n(X) \to C_n(X)$ by $\rho(\sigma) = \epsilon_n \bar{\sigma}$.

claim ρ is a chain map, chain homotopic to the identity.

proof (assuming claim)

$$(\rho^* \phi \cup \rho^* \psi)(\sigma) = \phi(\epsilon_k \sigma|_{[v_k, \dots v_0]}) \psi(\epsilon_l \sigma|_{[v_{k+l}, \dots v_k]})$$

$$\rho^*(\psi \cup \phi)(\sigma) = \epsilon_{k+l} \psi(\sigma|_{[v_{k+l}, \dots, v_k]}) \phi(\sigma|_{[v_k, \dots, v_0]})$$

R commutative: $\epsilon_k \epsilon_l (\rho^* \phi \cup \rho^* \psi) = \epsilon_{k+l}(\psi \cup \phi)$

Exercise: $\epsilon_{k+l} = (-1)^{kl} \epsilon_k \epsilon_l$.

so $$\rho^* \phi \cup \rho^* \psi = (-1)^{kl} \rho^*(\psi \cup \phi)$$

ρ chain homotopic to identity $\Rightarrow$ $[\phi] \cup [\psi] = (-1)^{kl} [\psi] \cup [\phi]$ on homology classes.
$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

Proof of claim

recall: chain map $\partial \rho = \rho \partial$

$$\partial \rho(\sigma) = \epsilon_n \sum_{i=0}^{n} (-1)^i \sigma|_{[v_n, \dots, \hat{v}_{n-i}, \dots v_0]}$$

$$\rho \partial(\sigma) = \rho\left(\sum_{i=1}^{n} (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]}\right)$$