$\delta(\phi\cup\psi) = \delta\phi\cup\psi + (-1)^k\phi\cup\delta\psi$

check: (cocycle) $\cup$ (cocycle) = cocycle

cocycle $\cup$ coboundary = coboundary $\psi = \delta\alpha$ $\quad \delta(\phi\cup\alpha) = (-1)^k\phi\cup\delta\alpha = (-1)^k\phi\cup\psi.$

coboundary $\cup$ cocycle = coboundary $\phi = \delta\alpha$ $\quad \delta(\alpha\cup\psi) = \delta\alpha\cup\psi + 0 = \phi\cup\psi.$

this gives an induced product

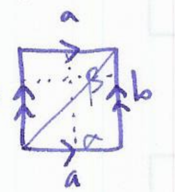
$$H^k(X;R)\times H^l(X;R) \longrightarrow H^{k+l}(X;R)$$
$$[\phi] \qquad [\psi] \longmapsto [\phi\cup\psi]$$

so $H^*(X;R)$ is a ring with identity $1\in H^0(X;R)$

$1 : \sigma \longmapsto 1$ for all $\sigma\in C_0(X)$.

Example

torus T^2 :



$$H^1(M)\times H^1(M) \longrightarrow H^2(M) \quad \text{(simplicial homology)}$$
$$\mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$$

basis for $H_1(M)$ consists of edges labelled a, b above (check; or use cellular homology)

$H^1(M) = \mathrm{Hom}(H_1(M);\mathbb{Z}) \cong \mathbb{Z}^2$ (universal coeffs).

choose dual basis α, β $\quad \alpha(a) = 1 \quad \beta(a) = 0$

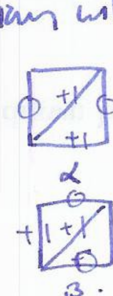
$\alpha(b) = 0 \quad \beta(b) = 0$

represent α by a cocycle ϕ: edges to $\mathbb{Z}$ s.t. $\delta\phi = 0$

think: intersections with arc α

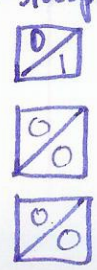
represent β by a cocycle ψ giving intersections with arc β.

compute $\alpha\cup\beta$: 2-simplices to $\mathbb{Z}$



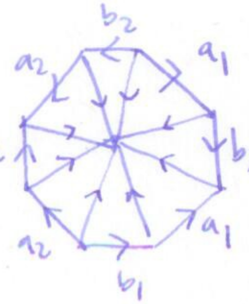
note: $H^2(M)\cong\mathbb{Z}$ generated by one copy of each simplex.

$\alpha\cup\beta$
$\alpha\cup\alpha$
$\beta\cup\beta$



Example M closed orientable surface of genus g $\quad H^1(M)\times H^1(M)\to H^2(M)$

$$\mathbb{Z}^{2g}\times\mathbb{Z}^{2g}\to\mathbb{Z}$$



basis for $H_1(M)$ consists of edges labelled a_i, b_i.

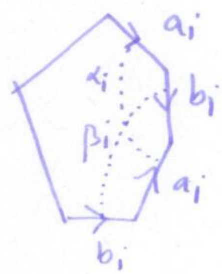
$$\overset{\cong}{\mathrm{Hom}}(H_1(M),\mathbb{Z})\cong\mathbb{Z}^{2g}$$

choose dual basis $\quad \alpha_i: \quad \alpha_i(a_j)=\delta_{ij}$

$$\alpha_i(b_j)=0$$

$\beta_i \quad \beta_i(a_j)=0$

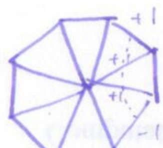
$$\beta_i(b_j)=\delta_{ij}.$$



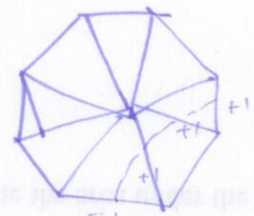
represent α_i by cocycle ϕ_i intersection with α_i

$\ \beta_i \qquad\qquad \psi_i \qquad\qquad\qquad \beta_i$

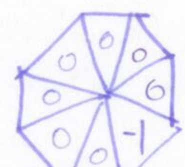
ϕ_1



ψ_1



$\phi_1\cup\psi_1$



claim $\quad \alpha_i\cup\beta_j = 1 = -\beta_j\cup\alpha_i \ \text{ if } i=j$

$\qquad\qquad 0$ otherwise.

recall $\quad (\phi\cup\psi)(\sigma)=\phi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$

gives $\quad H^k(X;R)\times H^l(X;R)\longrightarrow H^{k+l}(X;R)$

relative versions:

$$H^k(X;R)\times H^l(X,A;R)\xrightarrow{\cup}H^{k+l}(X,A;R)$$

$$H^k(X,A;R)\times H^l(X;R)\xrightarrow{\cup}H^{k+l}(X,A;R)$$

$$H^k(X,A;R)\times H^l(X,A;R)\xrightarrow{\cup}H^{k+l}(X,A;R)$$

as if either ϕ or $\psi=0$ on chains in A, then $\phi\cup\psi=0$ on chains in A.

Lemma $\quad H^k(X,A;R)\times H^l(X,B;R)\xrightarrow{\cup}H^{k+l}(X,A\cup B;R)$.

if A,B open in X, or subcomplexes of X if X is a CW-complex.

Proof: $\quad C^k(X,A;R)\times C^l(X,B;R)\xrightarrow{\cup}C^{k+l}(X,A+B;R)$

where $C^{k+l}(X, A+B; R)$ = ~~chains~~ cochains which vanish on chains which are sums of chains in A and chains in B. But $C^{k+l}(X, A+B; R) \hookrightarrow C^{k+l}(X, A\cup B; R)$ induces an isomorphism on cohomology by barycentric subdivision/excision, so this gives

$$H^k(X,A;R) \times H^l(X,B;R) \xrightarrow{\ \cup\ } H^{k+l}(X, A\cup B; R). \quad \square$$

__Proposition__ $f: X \to Y$ induces $f^*: H^n(Y;R) \to H^n(X;R)$ s.t. $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$, similarly for relative cohomology.

__Proof__ Write $f^\#$ for the cochain map: $f^\#(\phi)(\sigma) = \phi(f\sigma)$

suffices to show $f^\#(\phi) \cup f^\#(\psi) = f^\#(\phi \cup \psi)$.

$$\begin{aligned}(f^\#\phi \cup f^\#\psi)(\sigma) &= f^\#\phi\left(\sigma|_{[v_0,\dots,v_k]}\right) f^\#\psi\left(\sigma|_{[v_k,\dots,v_{k+l}]}\right)\\ &= \phi\left(f\sigma|_{[v_0,\dots v_k]}\right)\psi\left(f\sigma|_{[v_k,\dots v_{k+l}]}\right)\\ &= \phi\cup\psi(f\sigma) = f^\#(\phi\cup\psi) \quad \square\end{aligned}$$

We can now define the __cross product__ $H^k(X;R) \times H^l(Y;R) \xrightarrow{\times} H^{k+l}(X\times Y;R)$

(relative version: $H^k(X,A;R) \times H^l(Y,B;R) \xrightarrow{\times} H^{k+l}(X\times Y, A\times Y \cup X \times B; R)$)

$$\begin{array}{ccc} & X\times Y & \\ \swarrow p_1 & & \searrow p_2 \\ X & & Y \end{array} \quad \text{projections induce} \quad \begin{array}{ccc} & H^n(X\times Y) & \\ \uparrow p_1^* & & \uparrow p_2^* \\ H^n(X) & & H^n(Y) \end{array}$$

$$\begin{array}{ccc} H^k(X;R) \times H^l(Y;R) & \xrightarrow{\times} & H^{k+l}(X\times Y;R) \\ (\alpha\ ,\ \beta) & \longmapsto & p_1^*(\alpha) \cup p_2^*(\beta) \\ & & \alpha\times\beta \end{array}$$

__Example__ $T^n = S^1 \times S^1 \times \dots S^1$

recall $H_k(T^n;R) \cong R^{\binom{n}{k}}$ so by universal coefficients $H^k(T^n;R) \cong R^{\binom{n}{k}}$

__Fact__: all cohomology classes are products of 1-dimensional classes.

$H^*(T^2;\mathbb{Z})$: $\overset{0}{\mathbb{Z}}\ \ \overset{1}{\mathbb{Z}^2}\ \ \overset{2}{\mathbb{Z}}$ $\qquad H^*(T^3;\mathbb{Z})$: $\overset{0}{\mathbb{Z}}\ \ \overset{1}{\mathbb{Z}^3}\ \ \overset{2}{\mathbb{Z}^3}\ \ \overset{3}{\mathbb{Z}}$ $\qquad H^*(T^4;\mathbb{Z})$: $\overset{0}{\mathbb{Z}}\ \overset{1}{\mathbb{Z}^4}\ \overset{2}{\mathbb{Z}^6}\ \overset{3}{\mathbb{Z}^4}\ \overset{4}{\mathbb{Z}}$