<u>In general</u> let $\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$ be a chain complex.

dual chain complex: replace · C_n by the dual $\mathrm{Hom}(C_n, G) = C_n^*$

· ∂ by dual coboundary map δ

$$\delta = \partial^* : C_{n-1}^* \longrightarrow C_n^*$$

$$\begin{tikzcd} C_n \arrow[r, "\partial"] \arrow[d, "\partial\phi"'] & C_{n-1} \arrow[d, "\phi"] \\ G & G \end{tikzcd}$$

$$\delta(\phi) = \phi(\partial)$$

this gives a chain complex as $\partial\partial = 0 \Rightarrow \delta\delta = 0$.

$$\cdots \leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\partial} C_{n-1}^* \leftarrow \cdots$$

<u>Exercise</u>: $A \to B \to C \to 0$ exact $\Rightarrow$ $A^* \leftarrow B^* \leftarrow C^* \leftarrow 0$ exact.

<u>Thm</u> (Universal coefficients) If a chain complex C has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by split exact sequences

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C;G) \longrightarrow \mathrm{Hom}(H_n(C); G) \longrightarrow 0$$

split short exact sequence: $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$

a short exact sequence is <u>split</u> if $B \cong A \oplus C$ such that

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "a \mapsto (a{,}0)"'] & B \arrow[r, "j"] & C \arrow[r] & 0 \\ & & A \oplus C \arrow[ur, "(a{,}c) \mapsto c"'] \arrow[u] & & \end{tikzcd}$$

commutes.

<u>Exercises</u>: $\Leftrightarrow$ there is $p : B \to A$ s.t. $pi = 1_A$

$\Leftrightarrow$ there is $s : C \to B$ s.t. $js = 1_C$.

Ext(H,G) (H, G abelian groups)

Properties $\mathrm{Ext}(H\oplus H', G) \cong \cancel{H}\,\mathrm{Ext}(H,G)\oplus \mathrm{Ext}(H',G)$

$\mathrm{Ext}(H,G)=0$ if H is free.

$\mathrm{Ext}(\mathbb{Z}_n, G)\cong G/n\mathbb{Z}$.

Examples $\mathrm{Ext}(G,\mathbb{Z}) = \mathrm{Tor}(G)$

$\mathrm{Hom}(G,\mathbb{Z}) = \mathrm{Free}(G)$.

Let H be an abelian group. A __free resolution__ is

$$0 \to F_1 \to F_0 \to H \to 0 \quad \text{exact}$$

F_1 ↑ relations. F_0 ↑ generators

example $H=\mathbb{Z}\oplus\mathbb{Z}_3$.

$$0\to \mathbb{Z}\to \mathbb{Z}\oplus\mathbb{Z}\to \mathbb{Z}\oplus\mathbb{Z}_3\to 0$$
$$1\mapsto (0,3)$$

dual:

$$0\leftarrow \mathrm{Hom}(F_1;G)\leftarrow \mathrm{Hom}(F_0,G)\leftarrow \mathrm{Hom}(H,G)\leftarrow 0.$$

__not__ exact! at $\mathrm{Hom}(F_1;G)$.

Defn $\mathrm{Ext}(H,G) = H^1(F;G)$

Thm This is well defined $\square$.

Interpretations $\mathrm{Ext}(H,G)$ = isomorphism classes of extensions

$$0\to G\to J\to H\to 0.$$

Corollary if the homology groups H_n, H_{n-1} of a chain complex C are finitely generated, with torsion subgroups $T_n\subset H_n$ $T_{n-1}\subset H_{n-1}$,

then $H^n(C;\mathbb{Z})\cong \left(H_n/T_n\right)\oplus T_{n-1}$

§ Cohomology of spaces

X space, G abelian group

$C^n(X;G)$ singular cochains with w/ coeffs in G

$C^n(X;G) = \mathrm{Hom}(C_n(X), G)$, $C_n(X)$ = singular n-chains.

so $\phi \in C^n(X;G)$ is a function from n-cochains to G

coboundary map $\delta : C^n(X;G) \to C^{n+1}(X;G)$

$$\delta\phi(\sigma) = \phi(\partial\sigma)$$
$$= \sum_i (-1)^i \phi(\sigma|_{[v_0, \dots, \hat{v}_i, \dots v_n]})$$

$\partial^2 = 0 \Rightarrow \delta^2 = 0.$

$\ker\delta$: cocycles

$\mathrm{im}\,\delta$: coboundaries

$$H^n(X;G) = \ker\delta / \mathrm{im}\,\delta = \text{cocycles}/\text{coboundaries}.$$

<u>recall</u>: $0 \to \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X;G) \to \mathrm{Hom}(H_n(X); G) \to 0$

so $H^0(X;G) \cong \mathrm{Hom}(H_0(X), G) \cong G^{\#\text{connected components}}$.

$H^1(X;G) \cong \mathrm{Hom}(H_1(X), G) \cong \mathrm{Hom}(\pi_1 X, G)$.

many things work in a similar way for cohomology.

- reduced cohomology $\tilde{H}^n(X;G)$
- relative cohomology groups $H^n(X,A;G)$
- long exact sequence of a pair

$$0 \leftarrow C^n(A) \xleftarrow{i^\#} C^n(X) \leftarrow C^n(X,A) \leftarrow 0 \quad \text{exact}.$$

gives $\cdots \leftarrow H^{n+1}(X,A) \xleftarrow{\delta} H^n(A) \xleftarrow{i^*} H^n(X) \xleftarrow{j^*} H^n(X,A) \xleftarrow{\delta} H^{n-1}(A) \leftarrow \cdots$

- induced homomorphisms $f : X \to Y$ $f^* : H^n(Y) \to H^n(X)$
- homotopy invariance : $f \simeq g \Rightarrow f^* = g^* : H^n(Y) \to H^n(X)$
- excision : $Z \subset A \subset X$ closure $(Z) \subset \operatorname{int}(A)$

  $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces $H^n(X; A; G) \xrightarrow[\cong]{i^*} H^n(X \setminus Z, A \setminus Z; G)$

- axioms for cohomology
- simplicial cohomology
- singular cohomology
- cellular cohomology
- Mayer-Vietoris sequence.

§3.2 Cup product

coefficients in a <u>ring</u> R : $\mathbb{Z}, \mathbb{Z}_n, \mathbb{Q}$ $\quad \cup : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$

$$\phi \qquad \psi \qquad \phi \cup \psi$$

we will define this at the chain level:

<u>Defn</u> $\quad \phi \cup \psi(\sigma) = \phi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_{k+1}, \dots, v_{k+\ell}]})$

<u>Lemma</u>: $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$

$\phi \in C^k(X;R)$
$\psi \in C^\ell(X;R)$ $\quad \phi \cup \psi \in C^{k+\ell}(X;R)$

<u>Proof</u> $\quad \delta\phi \cup \psi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_{k+1}]}) \psi(\sigma|_{[v_{k+1}, \dots, v_{k+\ell+1}]})$

$$(-1)^k \phi \cup \delta\psi = \sum_{i=k}^{k+\ell} (-1)^i \phi(\sigma|_{[v_0, \dots, v_k]}) \psi(\sigma|_{[v_k, \dots \hat{v}_i \dots v_{k+\ell+1}]})$$

add up: get $\phi \cup \psi(\partial\sigma) = \delta(\phi \cup \psi)$ $\square$.