Example the boundary map $H_n(X,A) \to H_{n-1}(A)$ in l.e.s of a pair.

$f:(X,A)\to(Y,B)$ gives

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_n(Y,B) \arrow[d, "\partial"] \\ H_{n-1}(A) \arrow[r, "f_*"] & H_{n-1}(B) \end{tikzcd}.$$

§2.C Simplicial approximation

Defn If K, L are simplicial complexes ~~(Δ-complexes, CW-complexes)~~ then a map $f: K\to L$ is simplicial if it sends each simplex of K to a simplex of L by a linear map taking vertices to vertices.

Thm If K is a finite simplicial complex and L is an arbitrary simplicial complex, then any map $f: K \to L$ is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of K.

Observation may have to subdivide, consider $f: S^1 \to S^1$ 

Defn σ simplex in simplicial complex X. The star of σ, $St(\sigma)$ is the union of all simplices containing σ. The open star of σ is the union of all interiors of simplices containing σ. $st(\sigma)$

Example  $st(\sigma)$ open and $\overline{st(\sigma)} = St(\sigma)$.

Proof Choose metric on K s.t. each simplex isometric to standard simplex in $\mathbb{R}^{n+1}$.

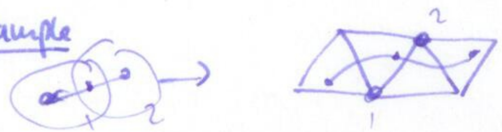
$\{st(v) \mid v \in L^0\}$ is an open cover of L, so $f^{-1}(st(v))$ is an open cover of K, with Lebesgue number ε. Subdivide K until diam of every simplex $< \varepsilon/2$

 homotop f to map each $v \in (K^{(r)})^0$ to some $w \in L$ s.t. $st(v) \subset f^{-1}(st(w))$.

 $\to$  by linear homotopies, and extend homotopy along simplices. $\square$.

Observation: if you subdivide L, can make homotopy arbitrarily small.

Example



For each $v \in K^0$ choose $w \in L^0$ s.t. ~~f(v~~ $st(v) \subset f^{-1}(st(w))$, defines $g: K^0 \to L^0$, extend to $g: K \to L$ by linear extension across simplices.

claim: $f \simeq g$: use linear homotopy as each $f(x), g(x)$ contained in a common simplex, as $st(v_1) \cap \dots \cap st(v_n) = \phi$, unless v_i are vertices of a common simplex σ, in which case $st(v_1) \cap \dots \cap st(v_n) = st(\sigma)$. $\square$.

Observation if you subdivide L, can make homotopy arbitrarily small.

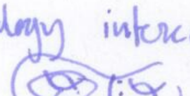
§3 Cohomology

formally: dualize homology

$$\begin{array}{ccccccc} \cdots \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow \cdots & \rightsquigarrow & H_n \\ & \downarrow & & \downarrow & & & \\ & \mathbb{Z} & & \mathbb{Z} & & & \end{array}$$

$$\cdots \leftarrow Hom(C_n, \mathbb{Z}) \xleftarrow{\delta} Hom(C_{n-1}, \mathbb{Z}) \leftarrow \cdots \quad \rightsquigarrow H^n$$

why bother? homology intersection form dual to cup product, which is easier to define at chain level



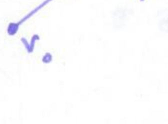
intuition. simplicial cohomology X: 2-dim simplicial complex

$\Delta^0(X;G)$: functions from X^0 to G

$\Delta^1(X;G)$ $\quad X^1 \quad G$

$\Delta^2(X;G)$ $\quad X^2 \quad G$

$$0 \leftarrow \Delta^2(X;G) \xleftarrow{\delta} \Delta^1(X;G) \xleftarrow{\delta} \Delta^0(X;G) \leftarrow 0$$

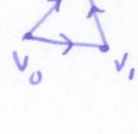


$\phi \in \Delta^0(X;G)$

$\delta\phi(e) = \phi(v_1) - \phi(v_0)$

$\quad \shortparallel$

$\phi(\partial e)$



$\psi \in \Delta^1(X;G)$

$\delta\psi(\sigma) = \psi(\partial\sigma)$

$= \psi([v_1, v_2]) - \psi([v_0, v_2]) + \psi([v_0, v_1])$

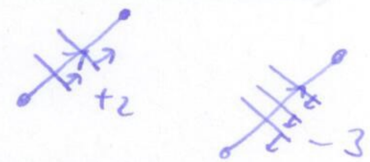
$H^0(X;G) = \ker(\delta) < \Delta^0(X;G)$

$\delta\phi = 0 \iff \phi(v_0) = \phi(v_1)$ for all vertices connected by an edge.

i.e. $H^0(X;G) \cong G^{\#\text{connected components of } X}$.

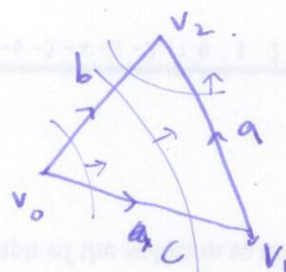
$H^1(X;G) = \ker(\delta)/\mathrm{im}(\delta)$

$\psi \in \Delta^1(X;G)$ assigns an integer to every edge.



$\delta\psi = 0$

"Locally trivial"



"level sets"

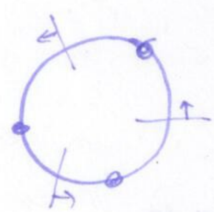
$\psi([v_1,v_2]) - \psi([v_0,v_2]) + \psi([v_0,v_1]) = 0$.

$a - b + c = 0$

"going around a loop which bounds gives zero".

$\mathrm{im}\,\delta$ = level sets coming from functions $\phi : X^0 \to \mathbb{Z}$.

Example



$\delta\psi = 0$, but $\psi \neq \delta\phi$ for any $\phi : X^0 \to \mathbb{Z}$.

§3.1 Cohomology groups

Universal coeffs.

Example $\quad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

H_*: $\quad \mathbb{Z} \quad 0 \quad \mathbb{Z}_2 \quad \mathbb{Z}$

dualize: $\quad 0 \leftarrow \mathrm{Hom}(\mathbb{Z};\mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow 0$

$0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$

H^* $\quad \mathbb{Z} \quad \mathbb{Z}_2 \quad 0 \quad \mathbb{Z}$